Module 7

M7.Strength and Failure Theories

M7.1 Strength of a Lamina

Learning Units of Module 8

M7.1 Strength of Laminates

M7.2 Failure Mechanics of Composites

M7.3 Macromechanical Failure Theories

M7.4 Comparison of Failure Theories

Definition of Strength of Lamina

- **"Effective Strength"** is defined as the ultimate value of the volume averaged stress which causes failure of a lamina under a simple state of stress.

Strength Terms

- Longitudinal Tension
- Longitudinal Compression
- Transverse Tension
- Transverse Compression

Definitions

Longitudinal means in the fiber direction or in the 1-direction

Transverse means perpendicular to the fibers or in the 2-direction

Strengths or Ultimate Stresses

$s_{L}^{(+)}$ *longitudinal tension*

$s_{L}^{(-)}$ *longitudinal compression*

$s_{T}^{(+)}$ *transverse tension*

$s_{T}^{(-)}$ *transverse compression*

s_{LT} *shear*

Corresponding Ultimate Strains

$e_L^{(+)}$ ***longitudinal tension***

$e_L^{(-)}$ ***longitudinal compression***

$e_T^{(+)}$ ***transverse tension***

$e_T^{(-)}$ ***transverse compression***

e_{LT} ***shear***

Assume Linear Behavior

$$s_L^{(+)} = E_1 e_L^{(+)}$$

$$s_L^{(-)} = E_1 e_L^{(-)}$$

$$s_T^{(+)} = E_2 e_T^{(+)}$$

$$s_T^{(-)} = E_2 e_T^{(-)}$$

$$s_{LT} = G_{12} e_{LT}$$

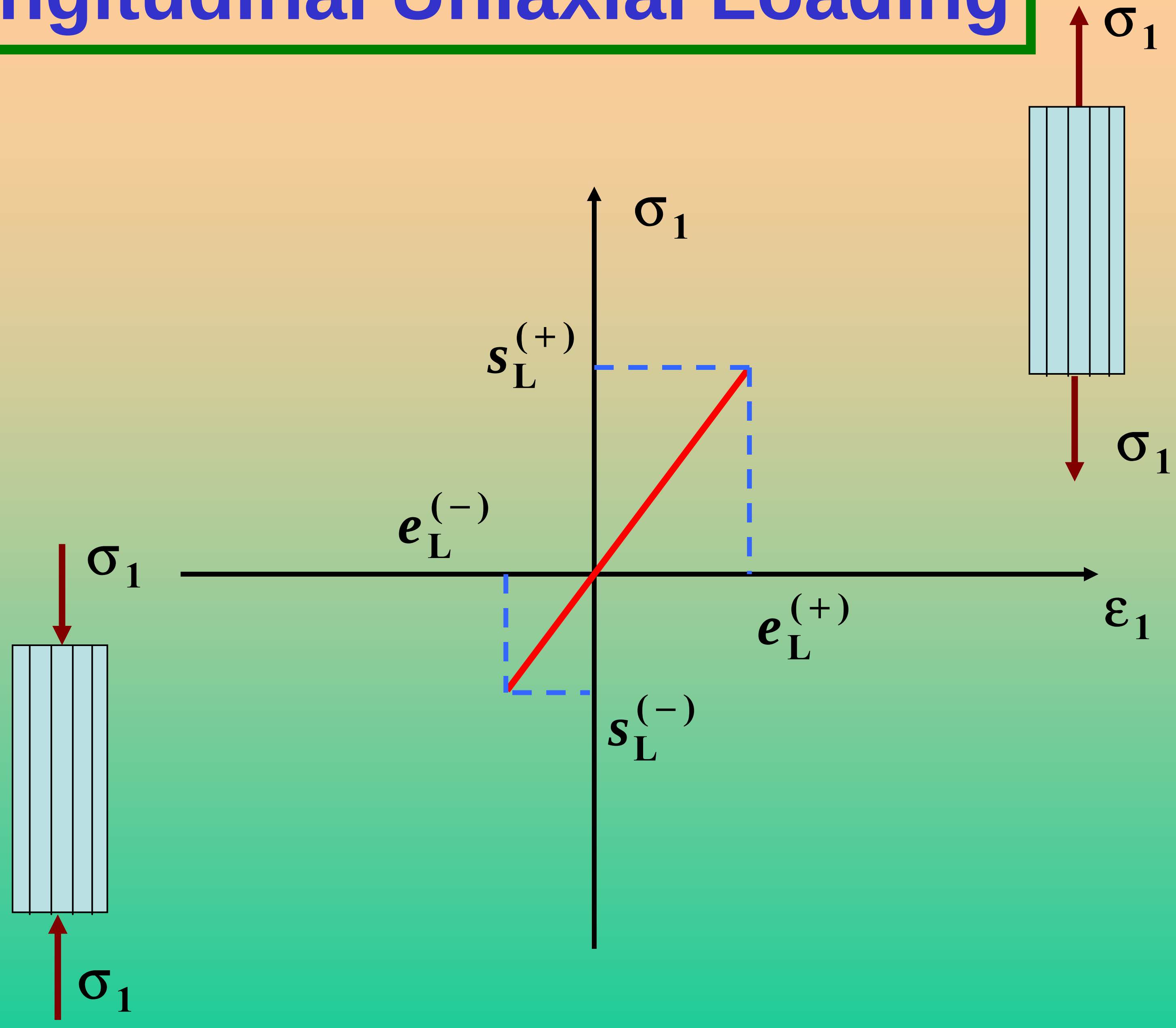
Multi-axial Stress State

- Off-axis or Multiaxial Loads
- Failure Criterion
- Semi-empirical
- Several Theories Proposed
- No One Universally Accepted Criterion

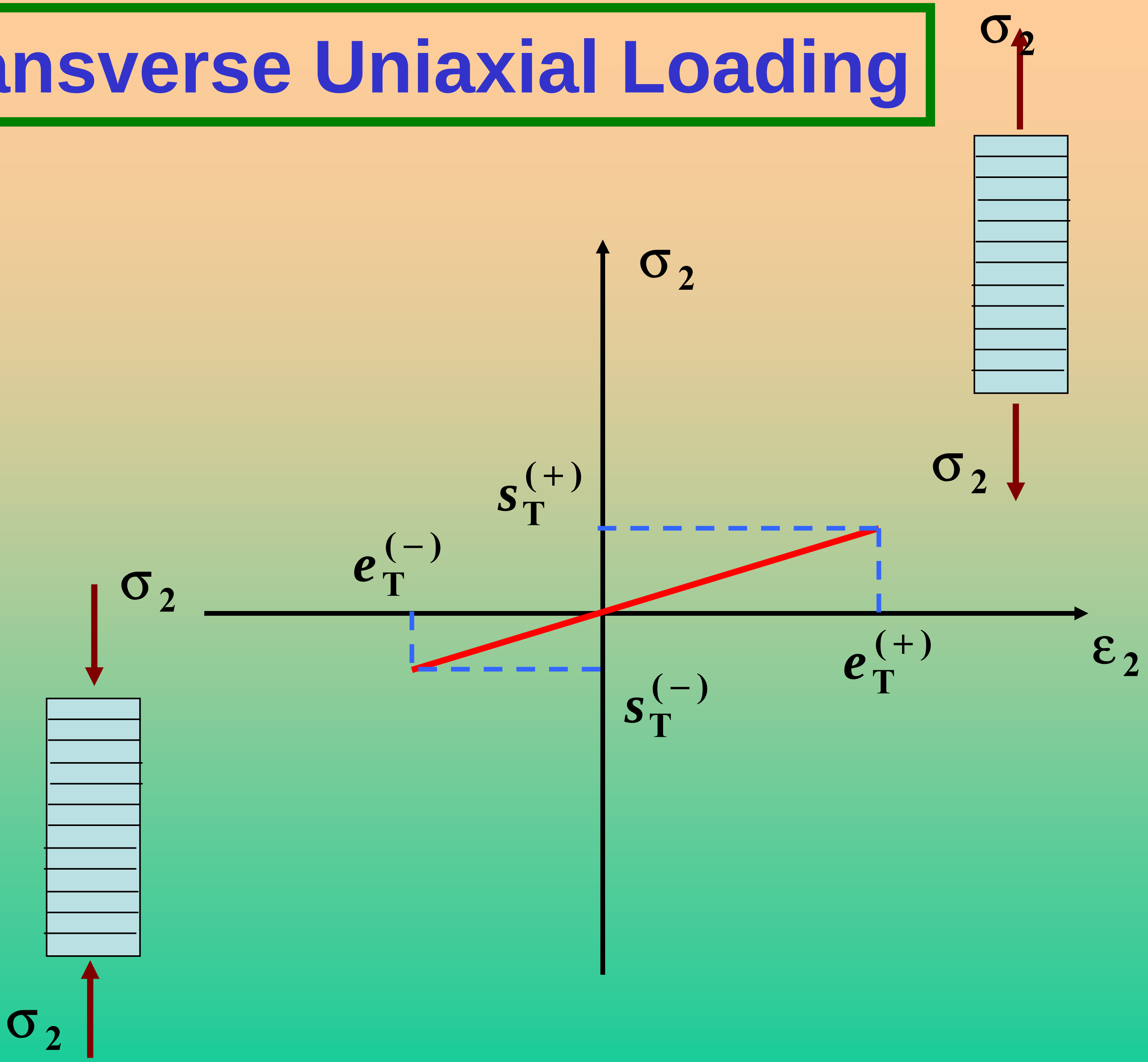
Strength Criteria

- Maximum Stress Theory
- Maximum Strain Theory
- Hill's Criterion
- Tsai-Hill Criterion
- Tsai-Wu Criterion

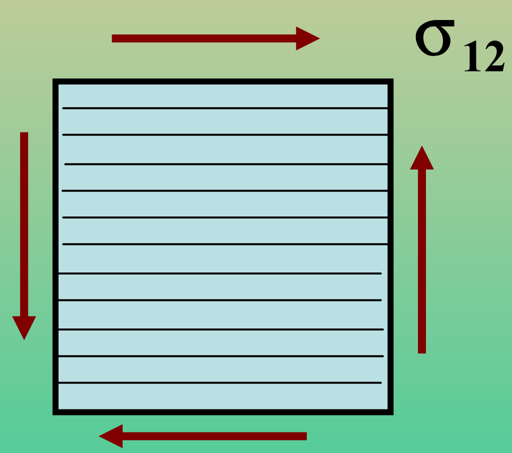
Longitudinal Uniaxial Loading

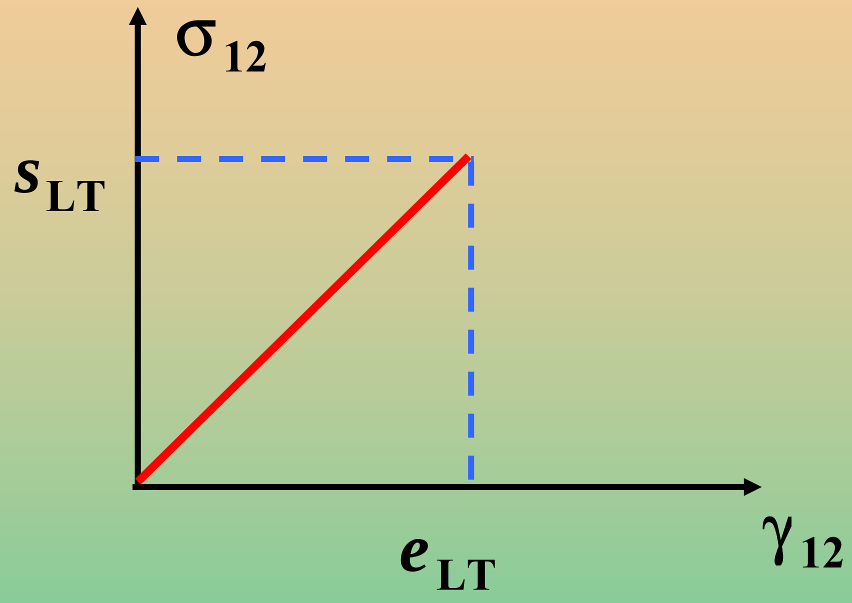


Transverse Uniaxial Loading



Shear Loading





Maximum Stress Theory

Predicts failure when any principal material axis stress component exceeds the corresponding strength:

$$\textit{No failure when:}$$

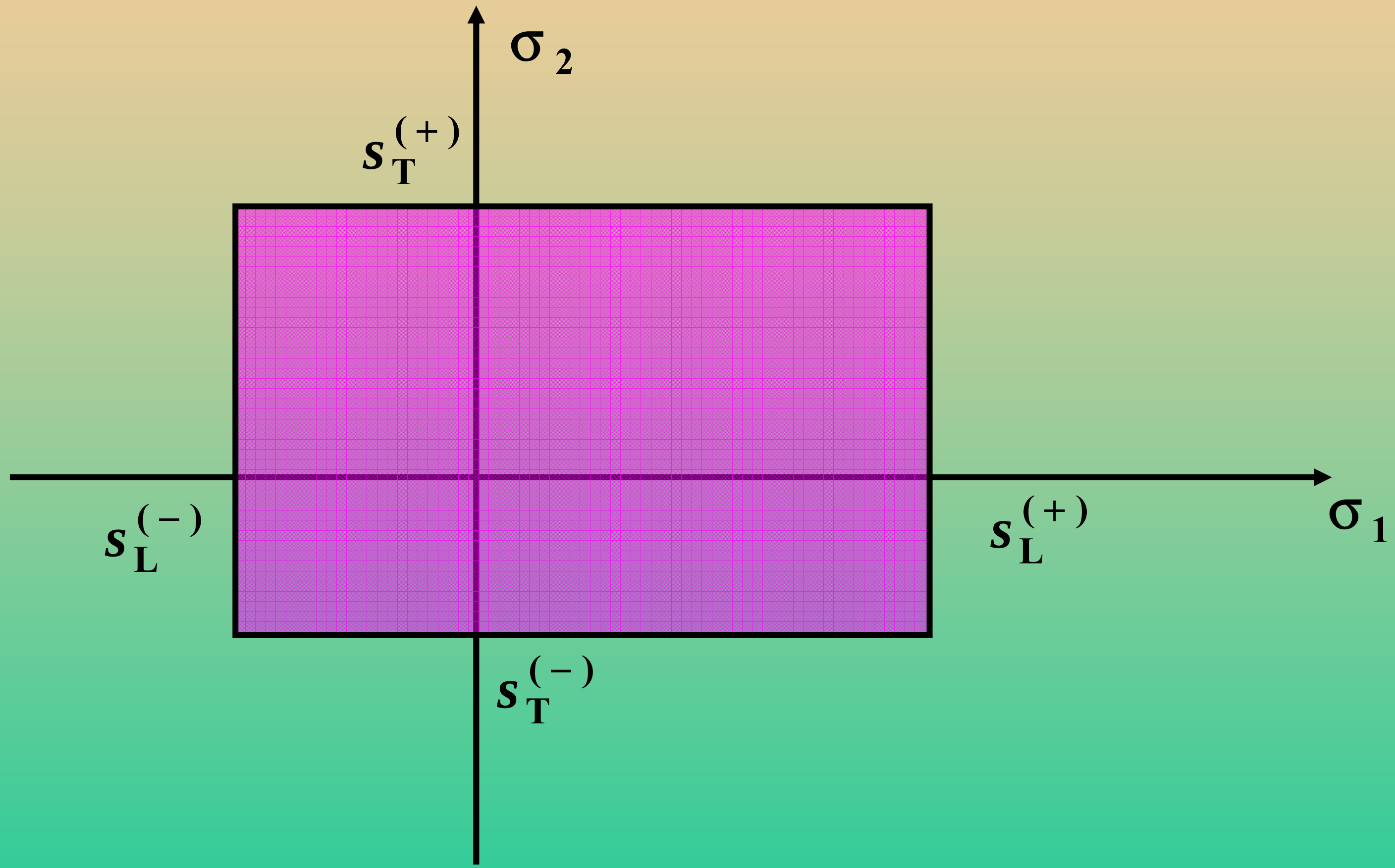
$$-s_L^{(-)} < \sigma_1 < s_L^{(+)}$$

$$-s_T^{(-)} < \sigma_2 < s_T^{(+)}$$

$$|\sigma_{12}| < s_{LT}$$

Failure Envelope

No interaction between stress terms.



Maximum Strain Theory

Predicts failure when any principal material axis strain component exceeds the corresponding ultimate strain:

$$\textit{No failure when:}$$

$$-e_L^{(-)} < \varepsilon_1 < e_L^{(+)}$$

$$-e_T^{(-)} < \varepsilon_2 < e_T^{(+)}$$

$$|\gamma_{12}| < e_{LT}$$

Stresses

$$\varepsilon_1 = \frac{s_L^{(+)}}{E_1} = \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_1}$$

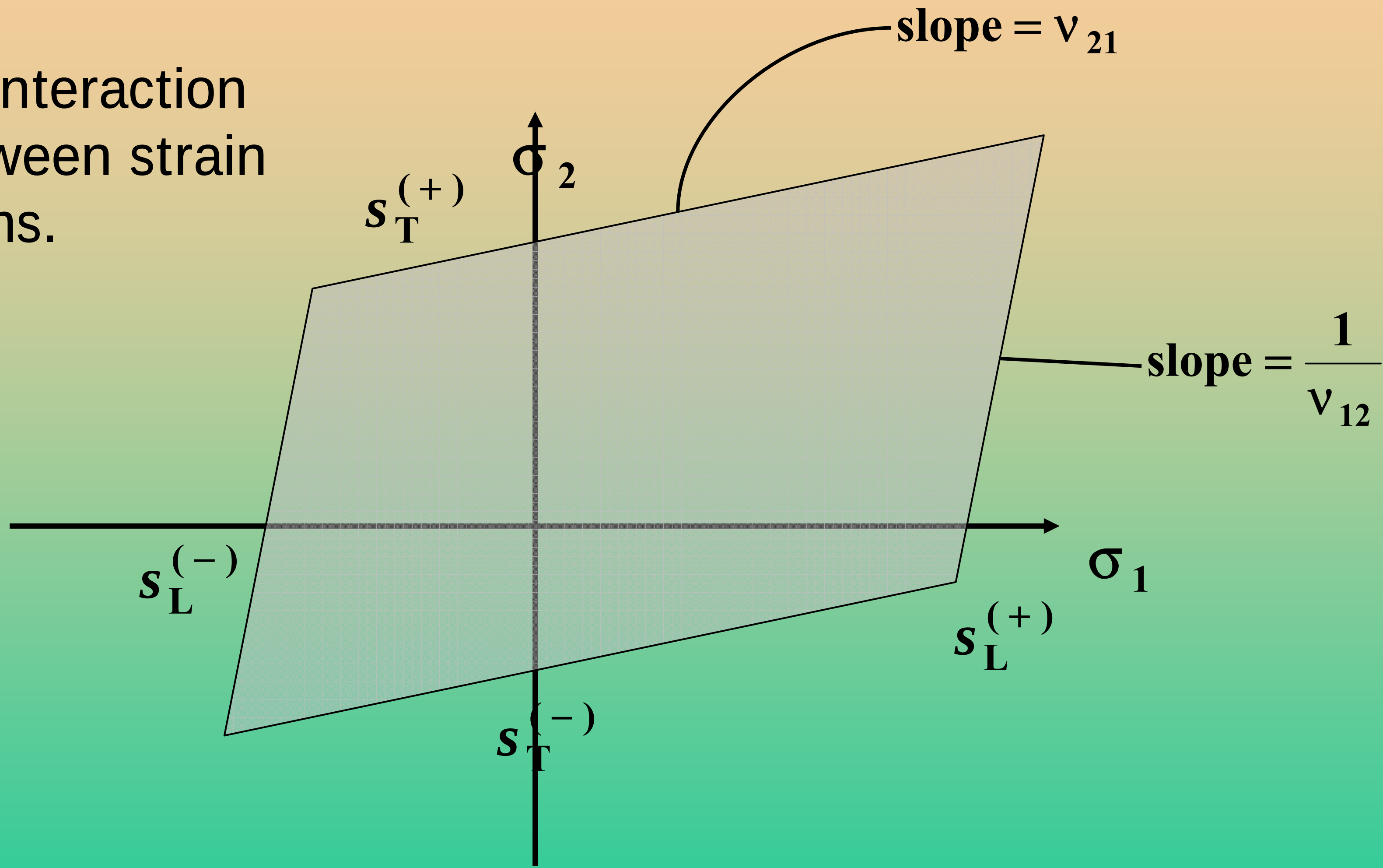
$$\sigma_2 = \frac{\sigma_1 - s_L^{(+)}}{\nu_{12}}$$

$$\varepsilon_2 = \frac{s_T^{(+)}}{E_2} = \frac{\sigma_2}{E_2} - \frac{\nu_{21}\sigma_1}{E_2}$$

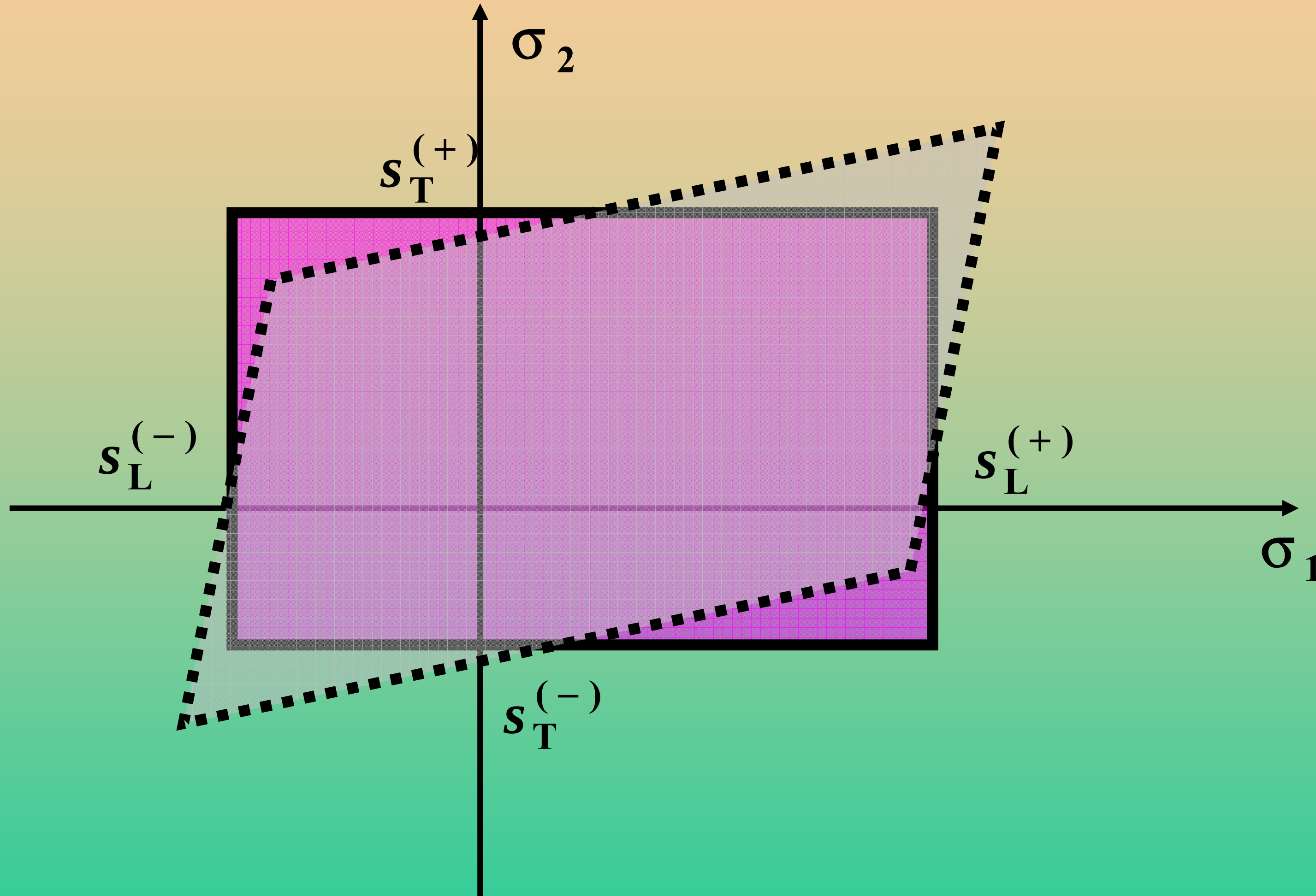
$$\sigma_2 = \nu_{21}\sigma_1 + s_L^{(+)}$$

Failure Envelope

No interaction between strain terms.



Comparison of Failure Envelopes



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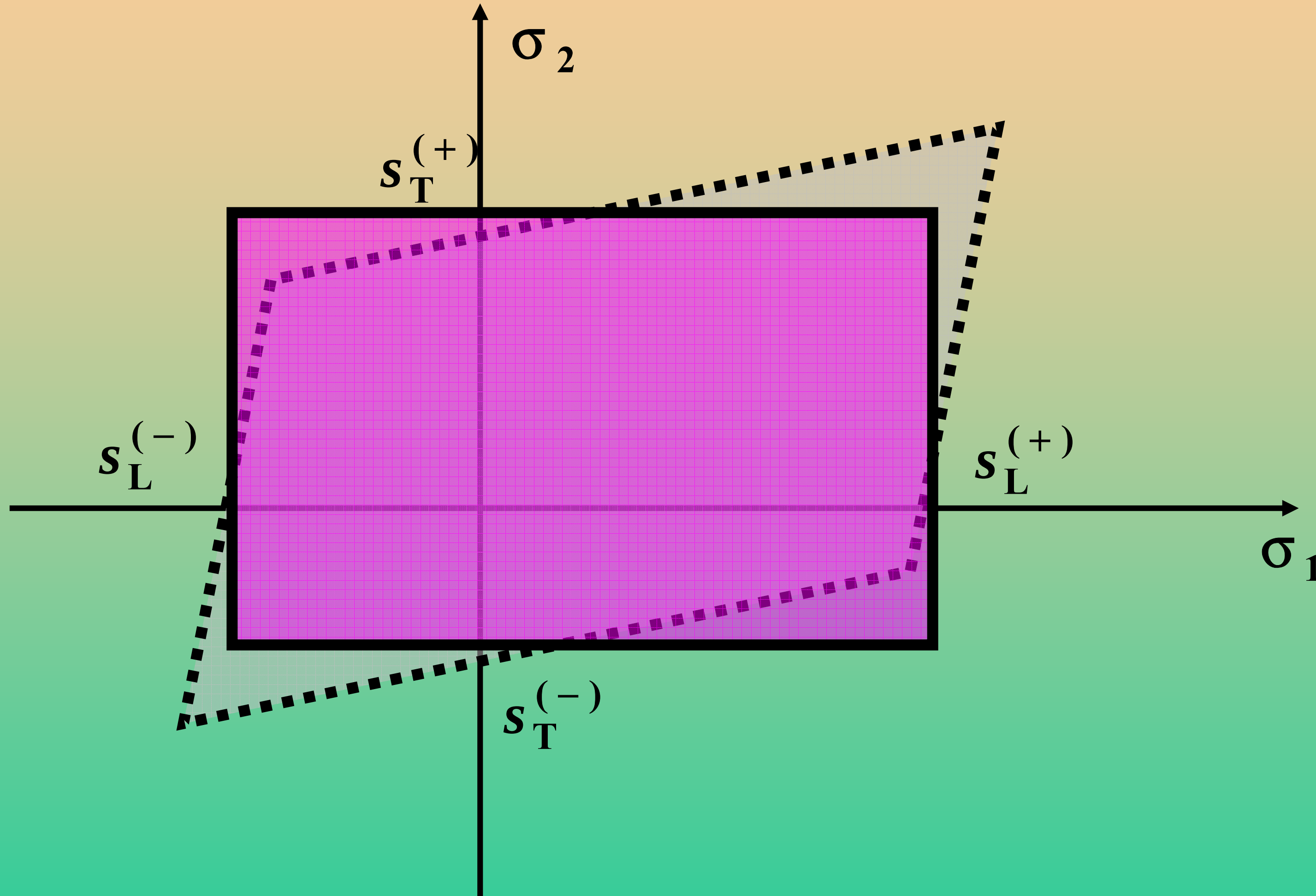


Hill's Criterion

Extension of von-Mises Criterion for isotropic materials.

3D equation :

$$\mathbf{A}(\sigma_2-\sigma_3)^2+\mathbf{B}(\sigma_3-\sigma_1)^2+\mathbf{C}(\sigma_1-\sigma_2)^2+$$
$$2\mathbf{D}\sigma_{23}^2+2\mathbf{E}\sigma_{31}^2+2\mathbf{F}\sigma_{12}^2\le 1$$

Continued...

Assume strengths in tension and compression are equal.

$$Y_1 = s_L^{(+)} = s_L^{(-)}$$

$$Y_2 = s_T^{(+)} = s_T^{(-)}$$

$$Y_{12} = s_{LT}$$

etc.

Hill's Criterion

Assume uniaxial loadings.

$$\mathbf{B} + \mathbf{C} = \frac{1}{\mathbf{Y}_1^2} \qquad \mathbf{A} + \mathbf{C} = \frac{1}{\mathbf{Y}_2^2}$$

$$\mathbf{A} + \mathbf{B} = \frac{1}{\mathbf{Y}_3^2}$$

Hill's Criterion Coefficients

$$2A = \frac{1}{Y_2^2} + \frac{1}{Y_3^2} - \frac{1}{Y_1^2} \qquad 2D = \frac{1}{Y_{23}^2}$$

$$2B = \frac{1}{Y_3^2} + \frac{1}{Y_1^2} - \frac{1}{Y_2^2} \qquad 2E = \frac{1}{Y_{31}^2}$$

$$2C = \frac{1}{Y_1^2} + \frac{1}{Y_2^2} - \frac{1}{Y_3^2} \qquad 2F = \frac{1}{Y_{12}^2}$$

2D Hill's Theory (Lamina)

- Assume plane stress
- Assume transverse isotropy

$$\sigma_3 = \sigma_{13} = \sigma_{23} = 0$$

$$Y_1 = s_L$$

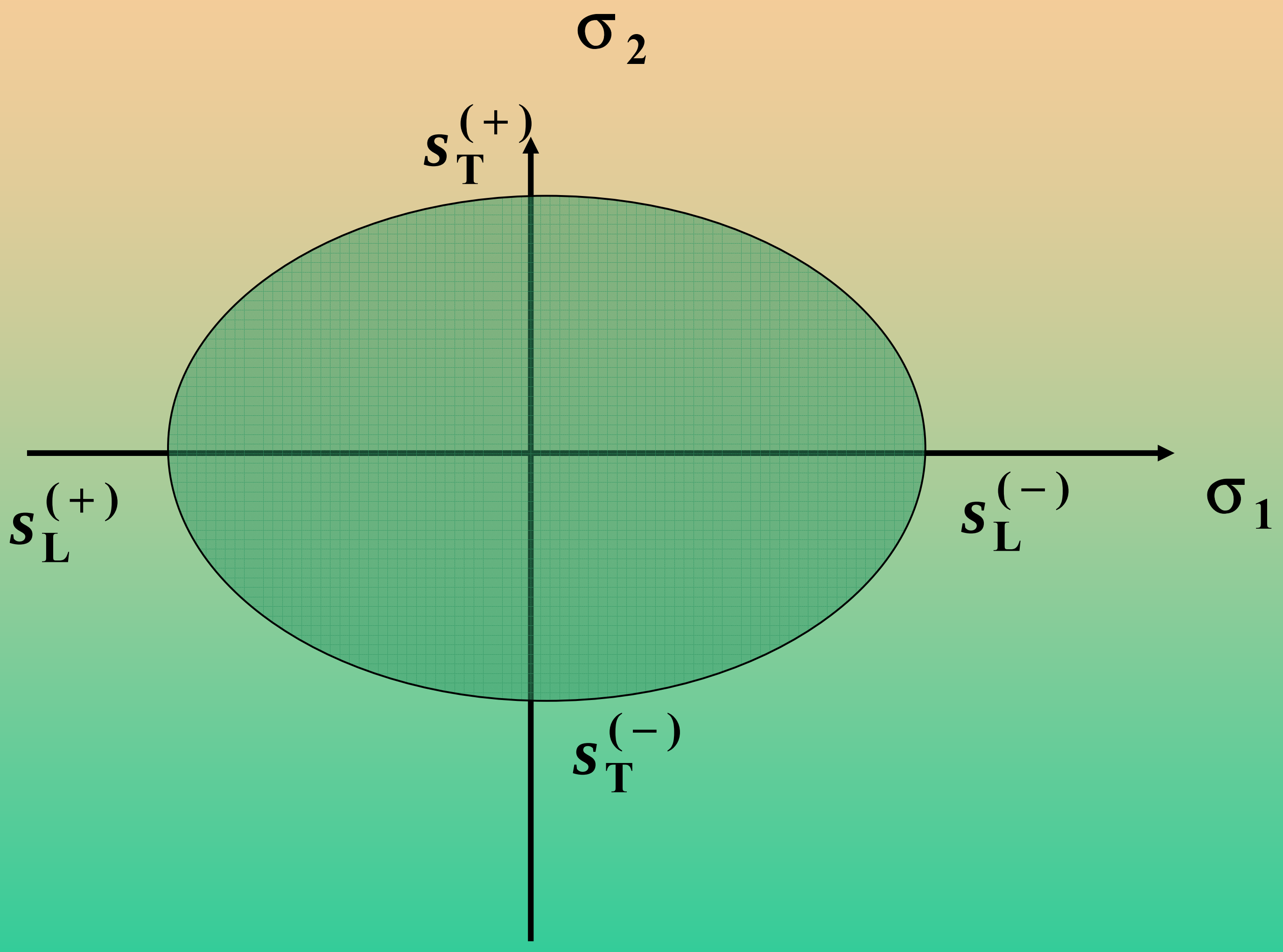
$$Y_2 = Y_3 = s_T$$

$$Y_{12} = s_{LT}$$

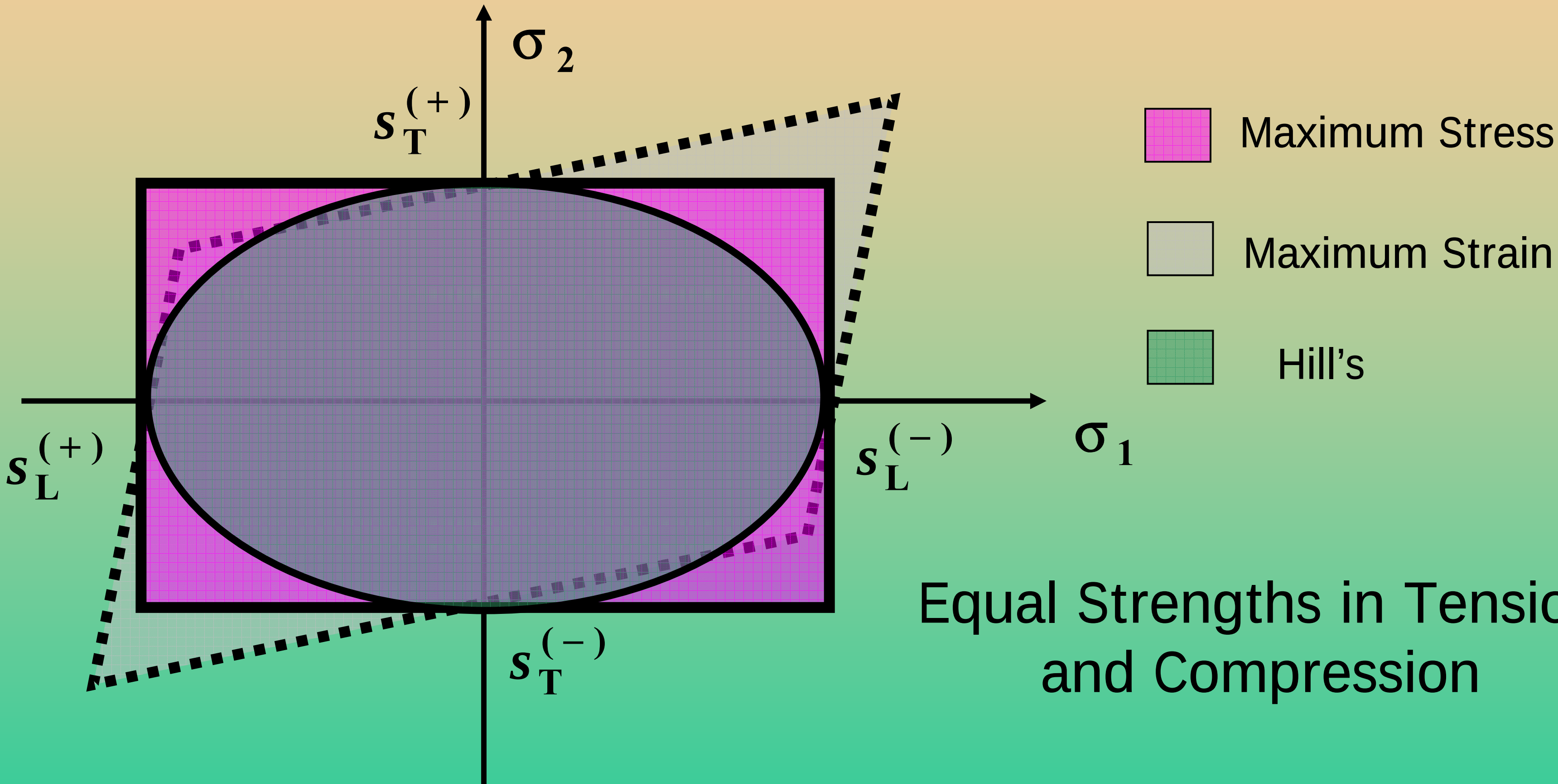
2D Hill's Theory (Lamina)

$$\frac{\sigma_1^2}{s_L^2} - \frac{\sigma_1 \sigma_2}{s_L^2} + \frac{\sigma_2^2}{s_T^2} + \frac{\sigma_{12}^2}{s_{LT}^2} \leq 1$$

Failure Envelope



Comparison of Failure Envelopes



Equal Strengths in Tension and Compression

Tsai-Hill's Criterion

$$\frac{\sigma_1^2}{s_L^2} - \frac{\sigma_1\sigma_2}{s_L^2} + \frac{\sigma_2^2}{s_T^2} + \frac{\sigma_{12}^2}{s_{LT}^2} \leq 1$$

$$s_L = s_l^{(+)} \qquad \sigma_1 > 0$$

$$s_L = s_l^{(-)} \qquad \sigma_1 < 0$$

$$s_T = s_l^{(+)} \qquad \sigma_2 > 0$$

$$s_T = s_l^{(-)} \qquad \sigma_2 < 0$$

Tsai-Hill's Criterion

$$F_i \sigma_i + F_{ij} \sigma_i \sigma_j \qquad i, j = 1, 2, \Lambda\ 6$$

F_i *and* F_{ij} ***strength tensors***

2D Tsai-Wu Theory

1. Assume plane stress
2. Assume transverse isotropy

$$\sigma_3 = \sigma_{13} = \sigma_{23} = 0$$

$$Y_1^T = s_L^{(+)} \qquad Y_1^C = s_L^{(-)}$$

$$Y_2^T = Y_3^T = s_T^{(+)} \qquad Y_2^C = Y_3^C = s_T^{(-)}$$

$$Y_{12} = s_{LT}$$

2D Tsai-Wu

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2$$
$$+ F_1\sigma_1 + F_2\sigma_2 + F_{12}\sigma_1\sigma_2 \leq 1$$

2D Tsai-Wu

Since failure is independent of the sign of $\sigma_6 \left(= \sigma_{12}\right)$

$$\Rightarrow \mathbf{F}_6 \equiv 0$$

Tsai-Wu Criterion (2D)

$$\mathbf{F}_{11} = \frac{1}{\boldsymbol{s}_{\mathbf{L}}^{(+)} \boldsymbol{s}_{\mathbf{L}}^{(-)}}$$

$$\mathbf{F}_{22} = \frac{1}{\boldsymbol{s}_{\mathbf{T}}^{(+)} \boldsymbol{s}_{\mathbf{T}}^{(-)}}$$

Tsai-Wu Criterion (2D)

$$F_1 = \frac{1}{s_L^{(+)}} - \frac{1}{s_L^{(-)}} = \frac{s_L^{(-)} - s_L^{(+)}}{s_L^{(+)} s_L^{(-)}}$$

$$F_2 = \frac{1}{s_T^{(+)}} - \frac{1}{s_T^{(-)}} = \frac{s_T^{(-)} - s_T^{(+)}}{s_T^{(+)} s_T^{(-)}}$$

$$F_{66} = \frac{1}{s_{LT}^2}$$

Quadratic Interaction Parameter

$$F_{12} = \textit{experimentaly determined parameter}$$

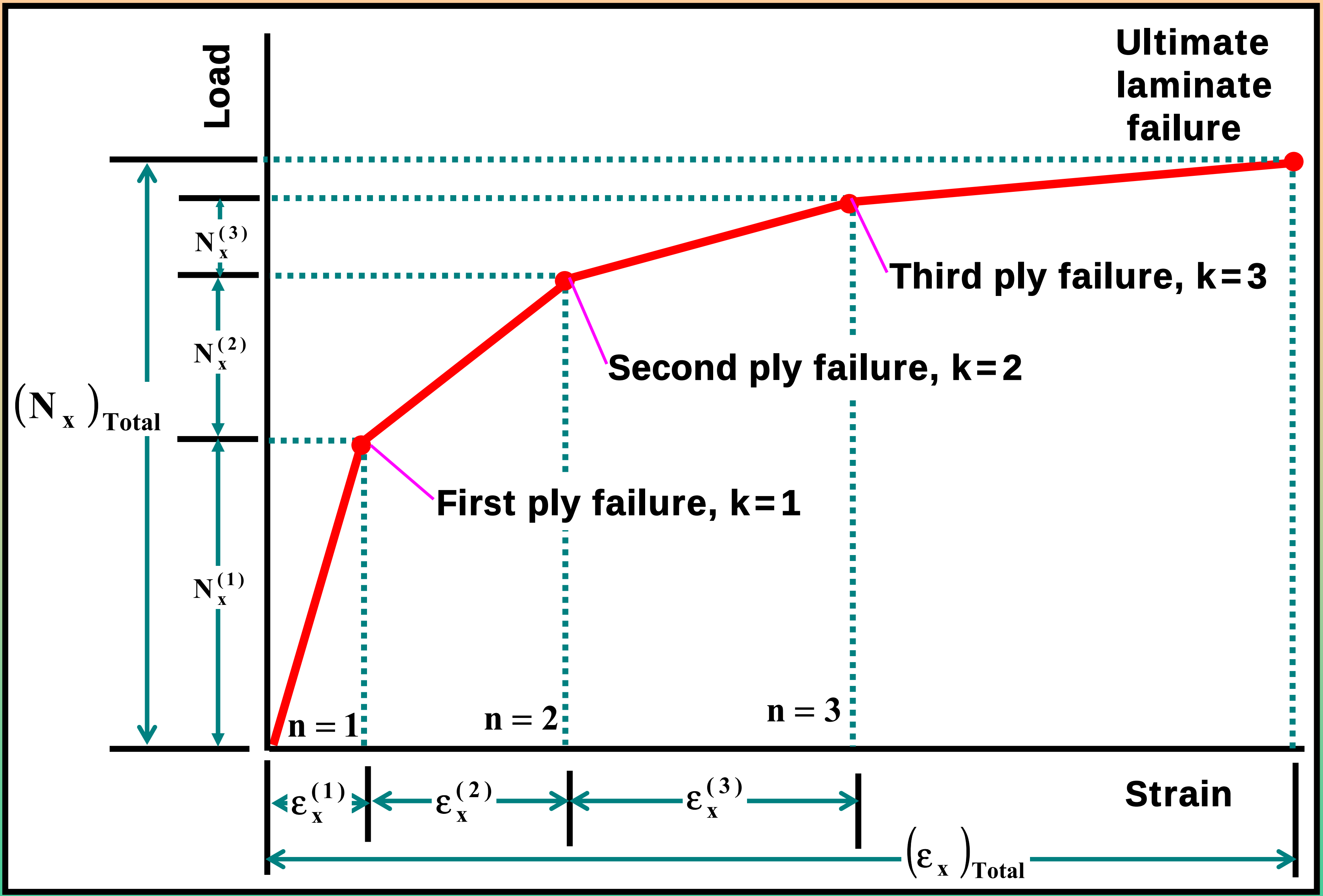
Tsai-Hahn Criterion

Quadratic Interaction Parameter :

$$\mathbf{F}_{12} \approx -\frac{\left(\mathbf{F}_{11}\mathbf{F}_{22}\right)^{\frac{1}{2}}}{2}$$

First-Ply-Failure

1. Load laminate elastically.
2. Calculate stresses and strains in each ply.
3. Apply the selected failure criterion to each ply.
4. Increase load until the first ply fails.
5. Model post failure behavior of failed ply.
6. Recalculate stiffness matrices and redistribute loads.
7. Continue to load till laminate failure.


Load
Ultimate laminate failure
$N_x^{(3)}$
Third ply failure, k=3
$N_x^{(2)}$
Second ply failure, k=2
$(N_x)_{Total}$
$N_x^{(1)}$
First ply failure, k=1
n = 1
n = 2
n = 3
$\varepsilon_x^{(1)}$
$\varepsilon_x^{(2)}$
$\varepsilon_x^{(3)}$
Strain
$(\varepsilon_x)_{Total}$

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix}_{\mathbf{Total}} = \sum_{k=1}^{n} \begin{Bmatrix} \mathbf{N}^{(n)} \\ \mathbf{M}^{(n)} \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{\varepsilon}^{0} \\ \boldsymbol{\kappa}^{0} \end{Bmatrix}_{\mathbf{Total}} = \sum_{k=1}^{n} \begin{Bmatrix} \boldsymbol{\varepsilon}^{0(n)} \\ \boldsymbol{\kappa}^{0(n)} \end{Bmatrix}$$

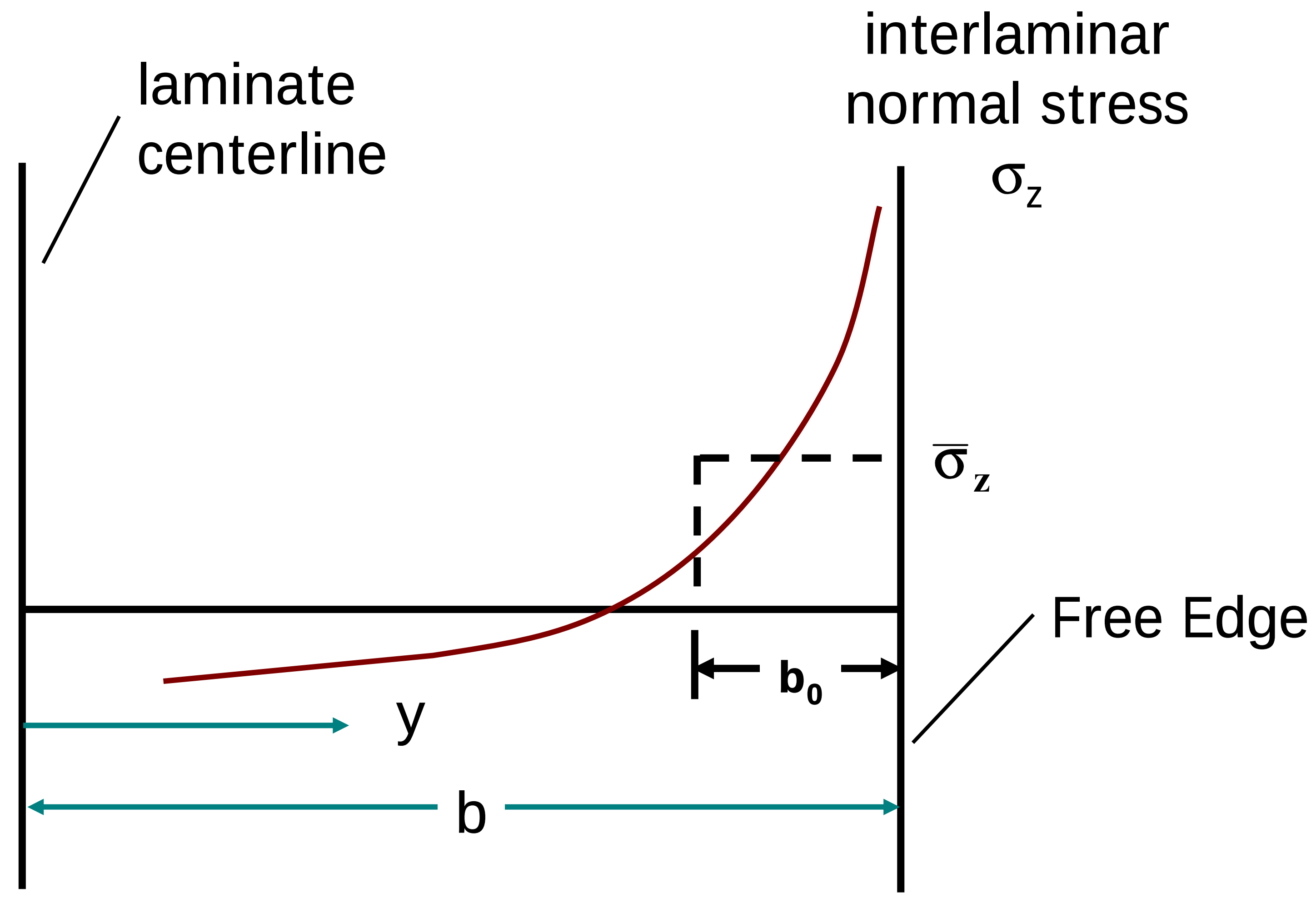
$$\begin{Bmatrix} \mathbf{N}^{(n)} \\ \hline \mathbf{M}^{(n)} \end{Bmatrix} = \left[\begin{array}{c:c} \mathbf{A}^{(n)} & \mathbf{B}^{(n)} \\ \hdashline \mathbf{B}^{(n)} & \mathbf{D}^{(n)} \end{array}\right] \begin{Bmatrix} \boldsymbol{\varepsilon}^{0(n)} \\ \hline \boldsymbol{\kappa}^{0(n)} \end{Bmatrix}$$

$A^{(n)}, B^{(n)}, D^{(n)}$ are the modified stiffnesses after the $(n-1)^{th}$ ply failure. They depend on $\left[Q^{(n)}\right]$, the modified ply stiffness matrix. This depends on the failure mode.

Simple Approach

If a ply fails set entire stiffness matrix equal to zero. Ply carries **NO LOAD** after failure. The load previously carried by the ply must be redistributed or picked up by the other plies. This redistribution may cause subsequent failure in additional plies. When the redistribution of load causes all plies to fail, the laminate is said to have failed.

Delamination


laminate
centerline
interlaminar
normal stress
σ_z
$\bar{\sigma}_z$
Free Edge
b_0
y
b

Kim-Soni Delamination Criterion

$$\bar{\sigma}_z = \frac{1}{b_0} \int_{b-b_0}^{b} \sigma_z(y,0)\,dy = s_Z^{(+)}$$

b_0 = one ply thickness

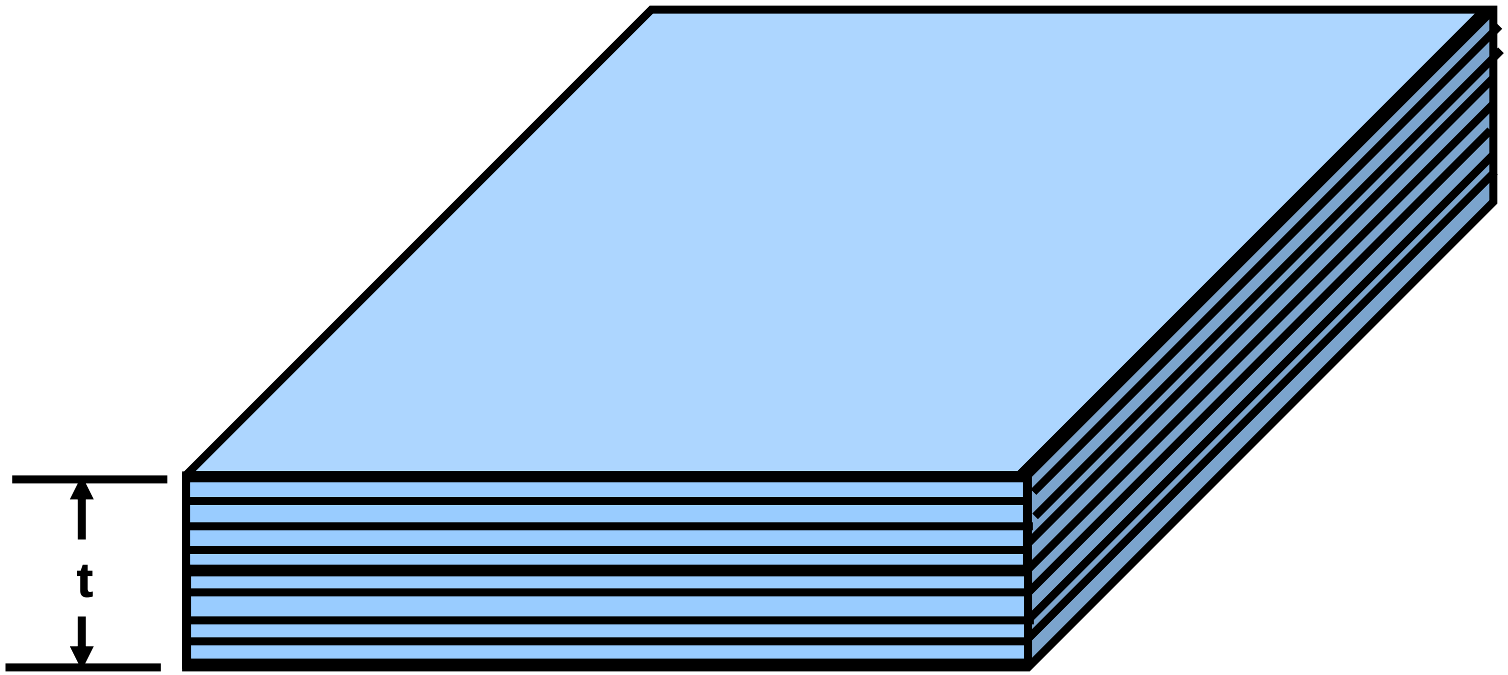
Quadratic Delamination Criterion

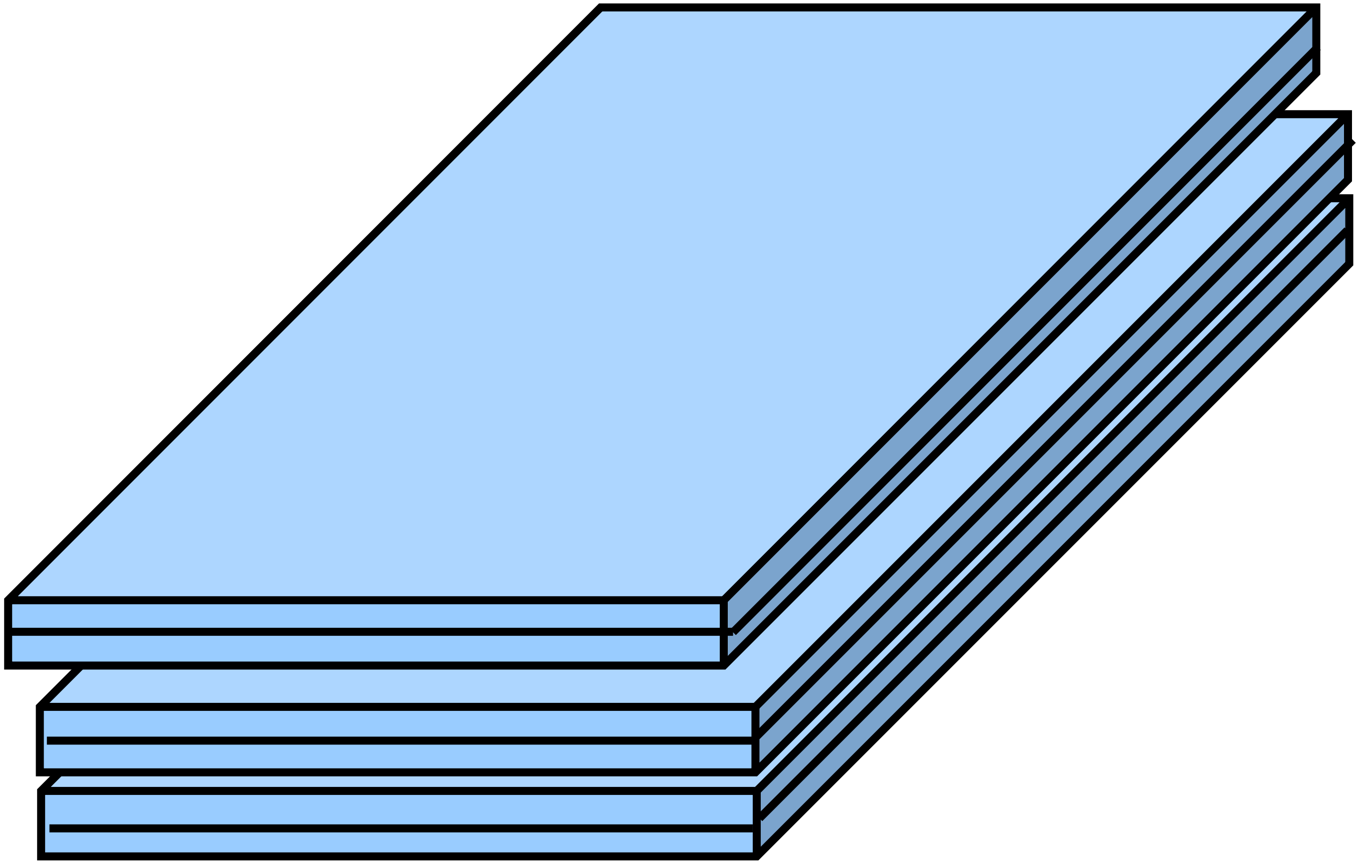
$$\left(\frac{\overline{\sigma}_{xz}}{\mathbf{S}_{xz}}\right)^2+\left(\frac{\overline{\sigma}_{yz}}{\mathbf{S}_{yz}}\right)^2+\left(\frac{\overline{\sigma}_z^{t}}{\mathbf{S}_z^{(+)}}\right)^2+\left(\frac{\overline{\sigma}_z^{c}}{\mathbf{S}_z^{(-)}}\right)^2=1$$

Simplified Quadratic Delamination Criterion

$$\left(\frac{\bar{\sigma}_{xz}}{S_{xz}}\right)^2 + \left(\frac{\bar{\sigma}_z^t}{S_z^{(+)}}\right)^2 = 1$$

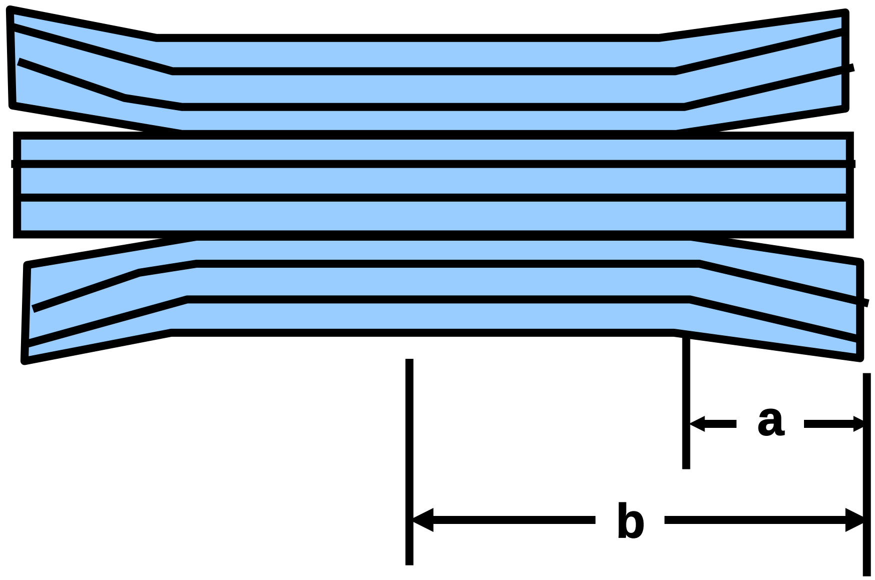
Laminated





Totally delaminated

Partially delaminated



Reduced Stiffness – Totally Delaminated

$$E_x = \frac{1}{tA'_{11}} \quad \text{effective modulus}$$

Totally Delaminated Specimen

$$E_{td} = \frac{\sum_{i=1}^{m} E_{xi} t_i}{t}$$

Reduced Stiffness

E_{td} $\begin{cases} \text{longitudinal Young's modulus of a} \\ \text{totally delaminated laminate} \end{cases}$

E_{xi} $\begin{cases} \text{longitudinal Young's modulus of the} \\ i^{th} \text{ sublaminate formed by the delamination} \end{cases}$

t_i $\{\text{thickness of } i^{th} \text{ sublaminate}$

m $\{\text{number of sublaminates}$

Reduced Stiffness – Partiality Delaminated

$$\mathbf{E} = \left(\mathbf{E}_{td} - \mathbf{E}_x\right)\frac{\mathbf{a}}{\mathbf{b}} + \mathbf{E}_x$$

$\mathbf{E}_{td}$ { **longitudinal Young's modulus of a partially delaminated laminate**

a { **delamination distance from free edge**

b { **half width of laminate**

Alternative Form – Partiality Delaminated

$$\mathbf{E} = \left(\mathbf{E}_{td} - \mathbf{E}_{x}\right)\frac{\mathbf{A}_{d}}{\mathbf{A}_{t}} + \mathbf{E}_{x}$$

$\mathbf{E}_{td}$ $\begin{cases}\text{longitudinal Young's modulus of a} \\ \text{partially delaminated laminate}\end{cases}$

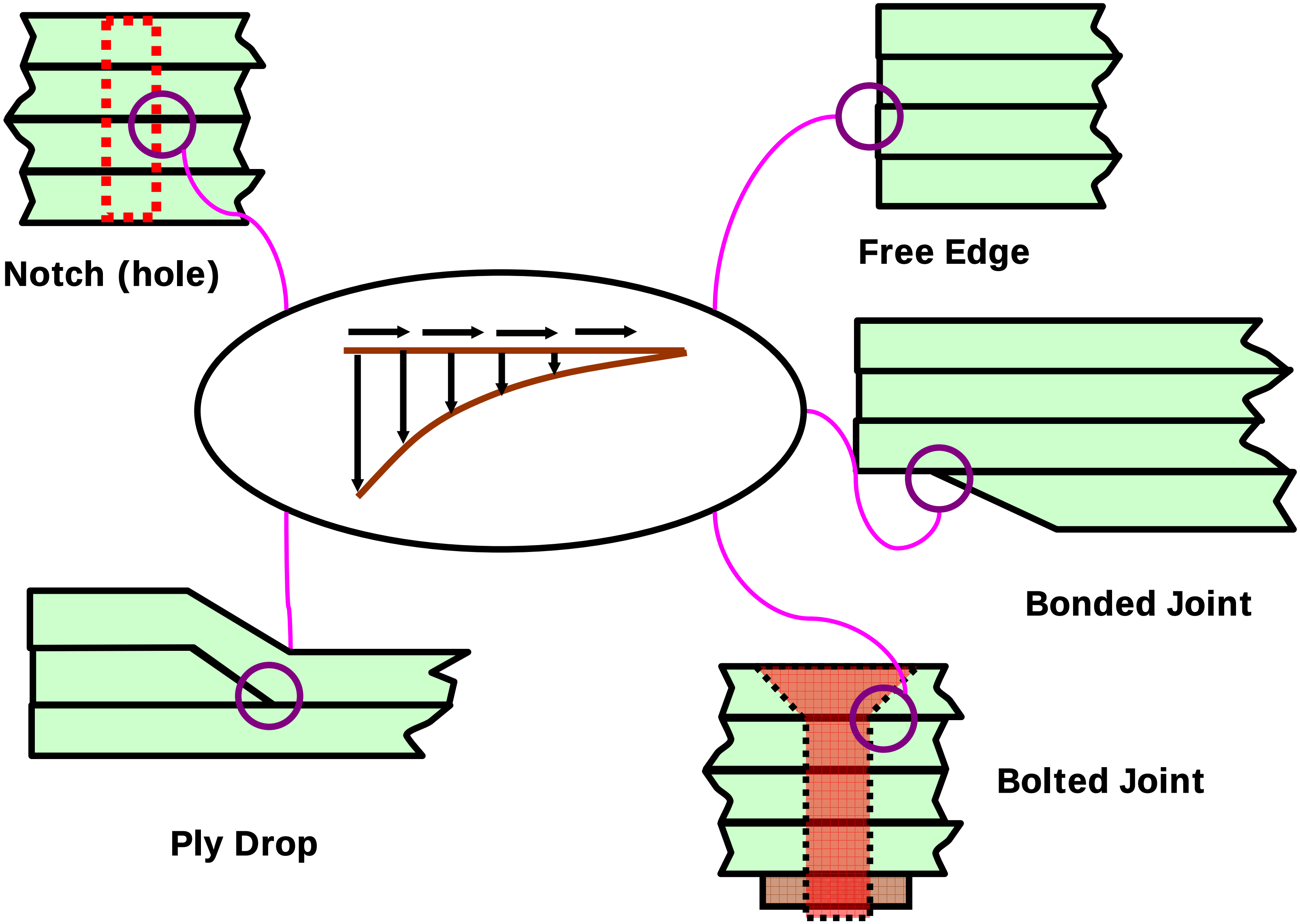
$\mathbf{A}_{d}$ $\{\text{delaminated area}$

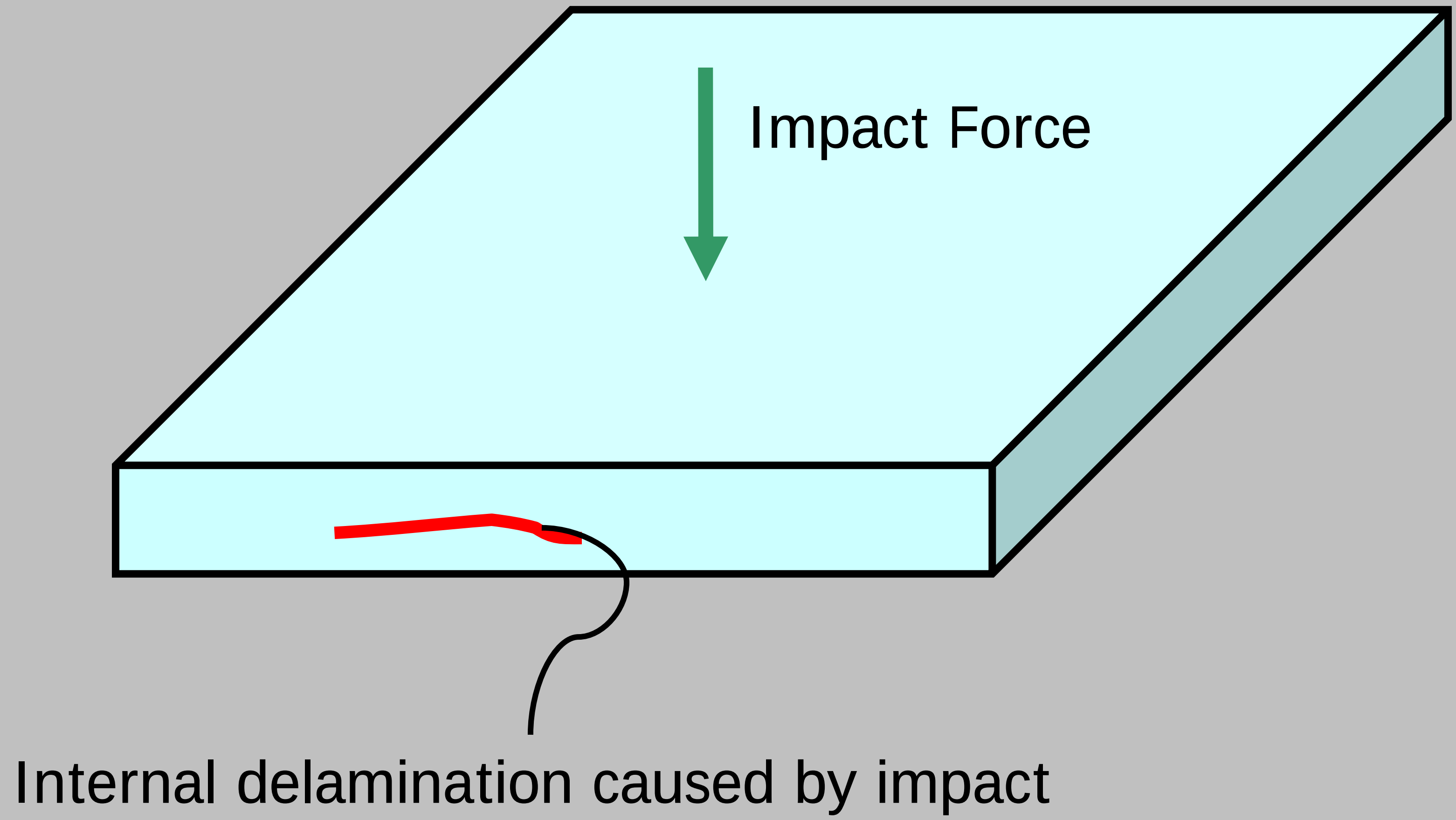
$\mathbf{A}_{t}$ $\{\text{total interfacial area}$

Sources of Delamination

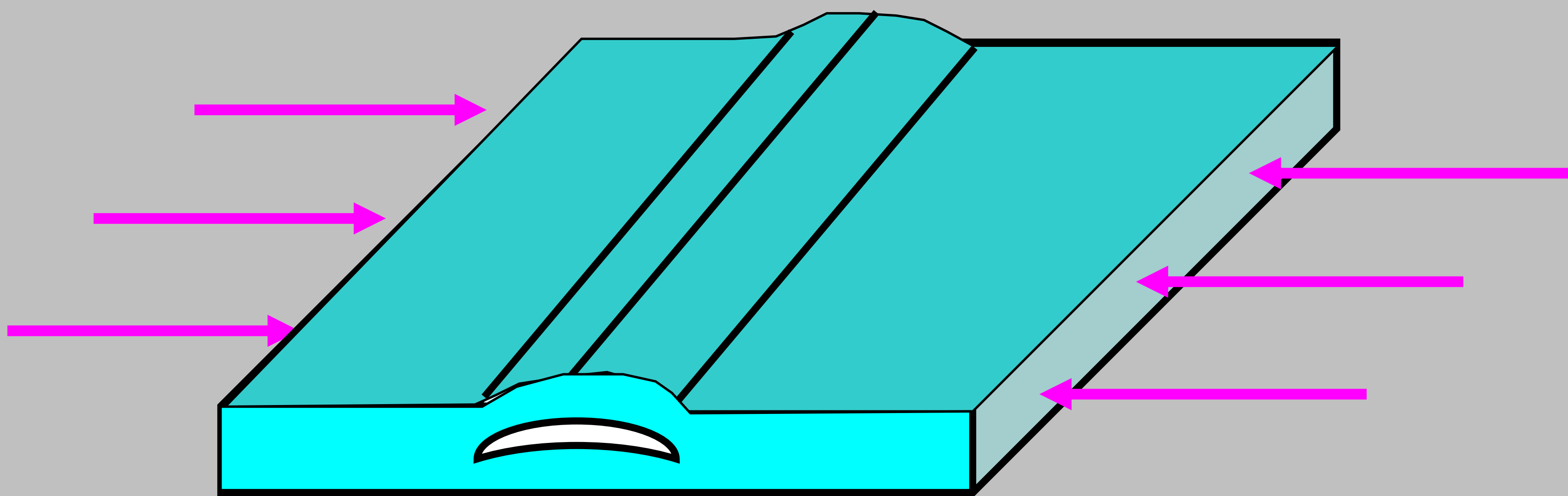
1. Free Edge
2. Notch or Hole
3. Ply Drop Off
4. Bond Joint
5. Bolted Joint
6. Foreign Object Damage (Low Velocity Impact)

DISCONTINUITIES




Impact Force
Internal delamination caused by impact

In-plane compressive loading after impact causes local buckling and reduction of compressive strength.



M7.2 Failure Mechanics of Composites

Introduction

- Failure in composites shows a much wider variety of mechanisms than that in metals. There are many different modes of failure. There are not a definite number of material's strength properties because it depends on the failure criterion adopted. However, for an orthotropic material, the following 9 strength properties are necessary (not always sufficient though), tensile and compressive strengths in the three principal directions of the material and shear strengths in the three directions.

Introduction

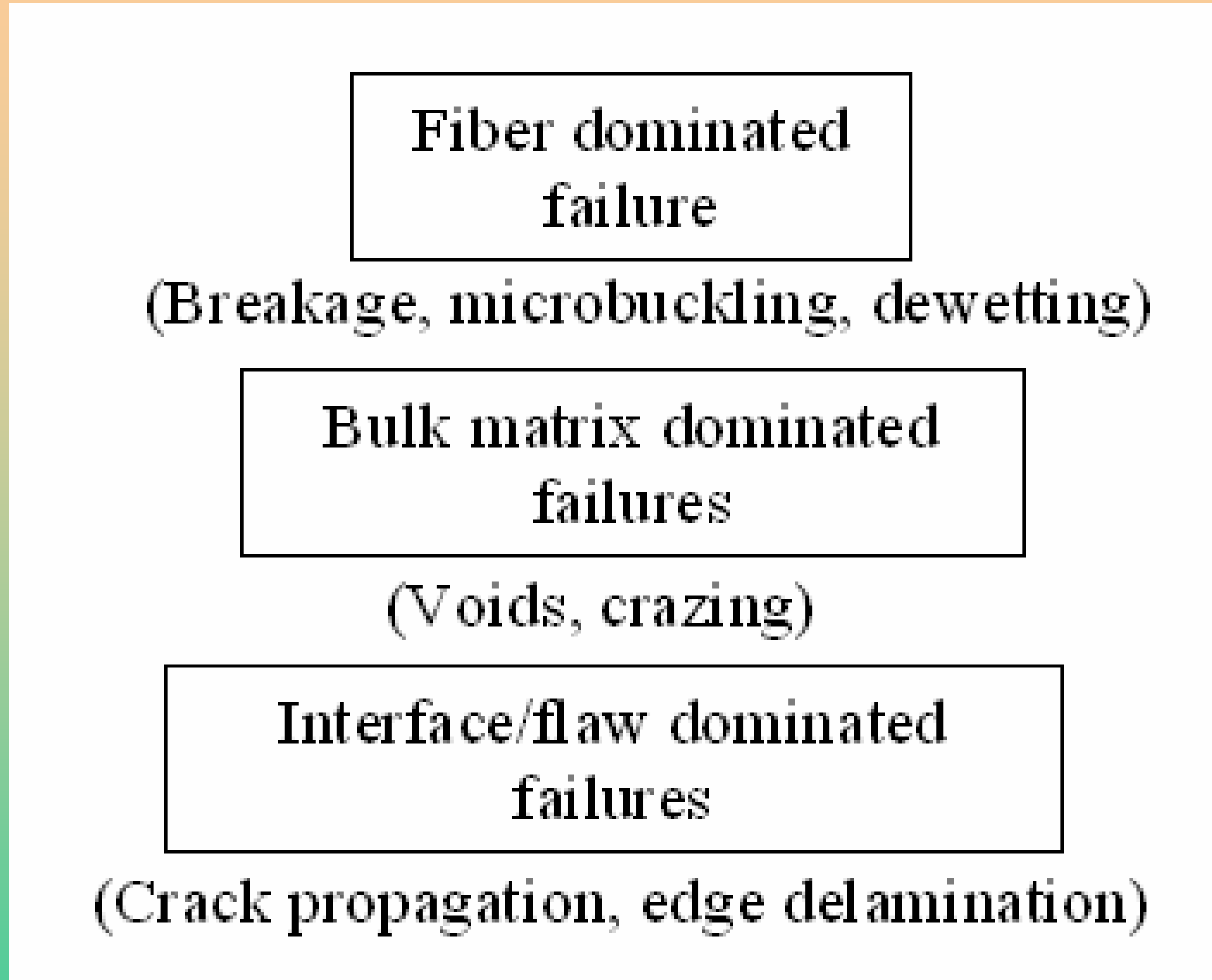


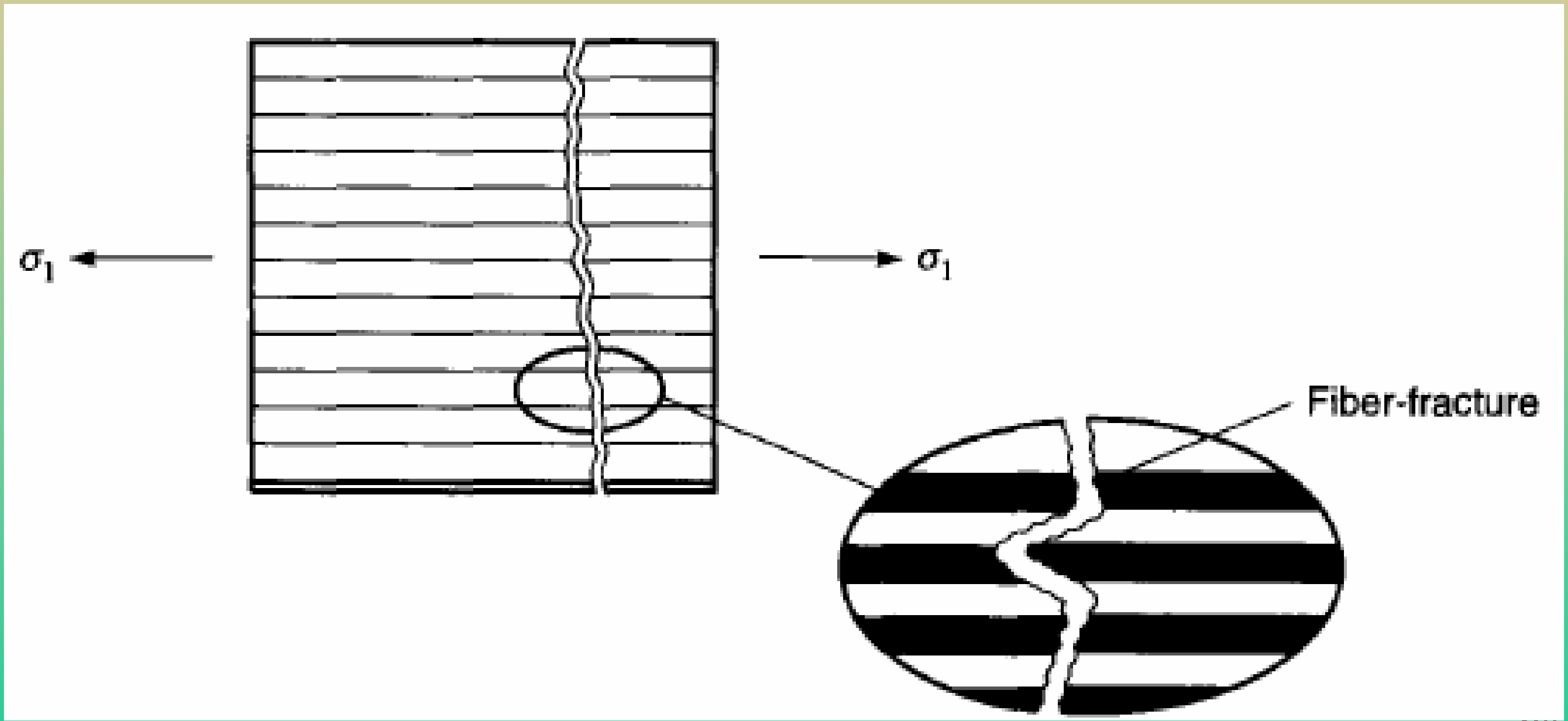
Figure: The different levels of failure characterization and disciplines linkage

M7.3 Macromechanical Failure Theories

M7.3 Macromechanical Failure Theories/Failure Theories for Fiber-Reinforced Materials

M7.3.1 Maximum Stress Criterion

Figure: Failure in tension in the 1 direction



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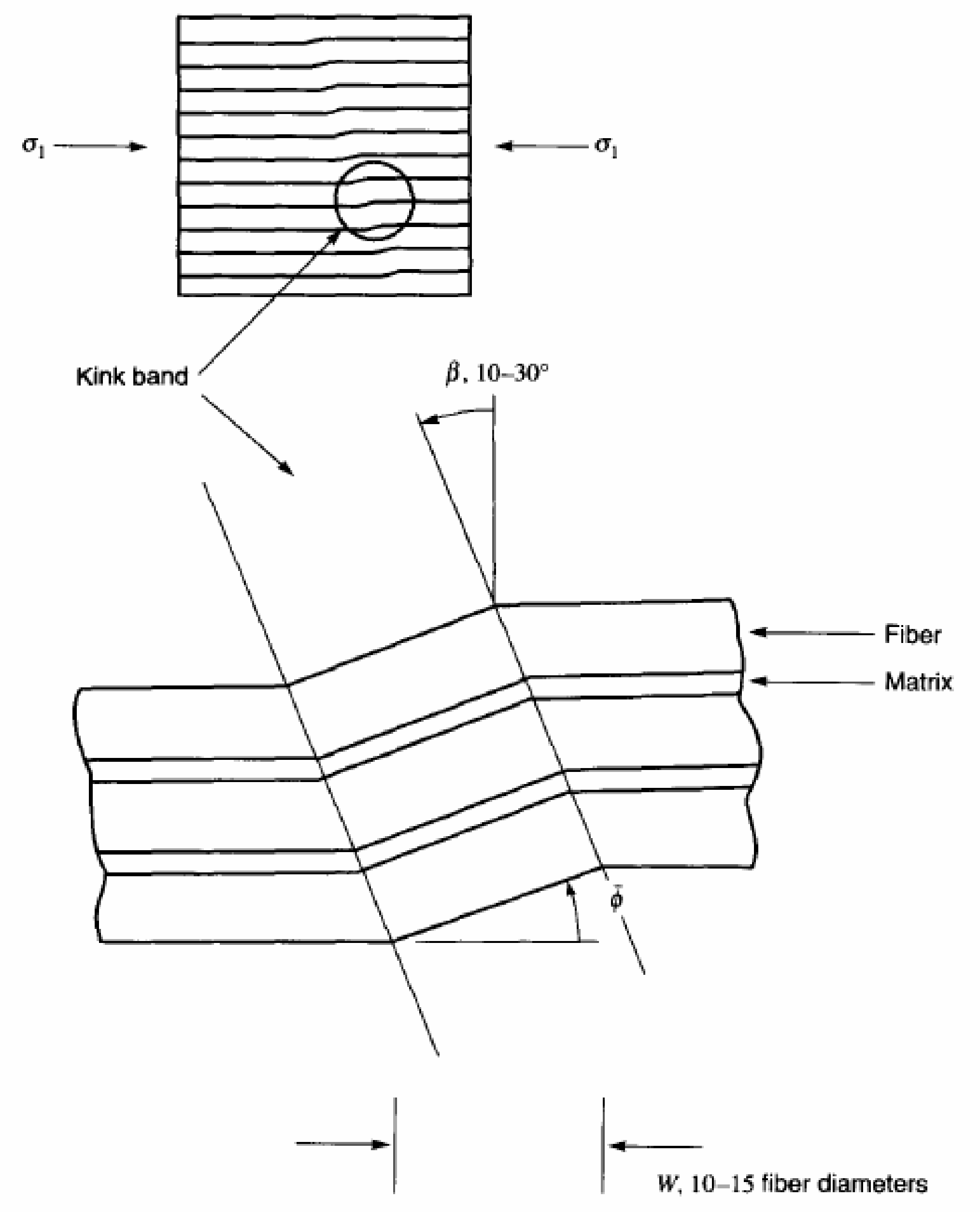
M7.3.1 Maximum Stress Criterion

Figure: Failure in compression in the 1 direction



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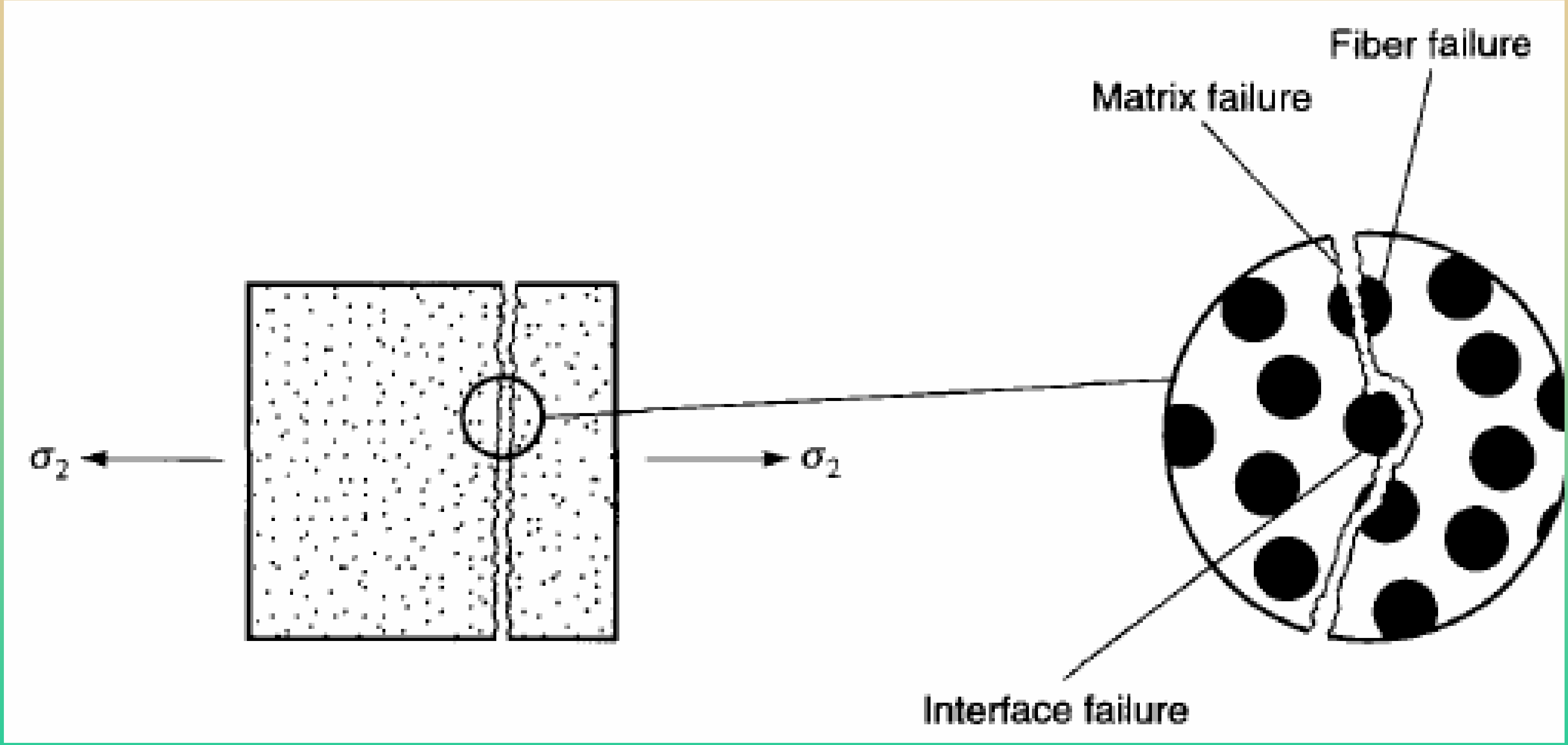
M7.3.1 Maximum Stress Criterion

Figure: Failure in tensile in the 2 direction



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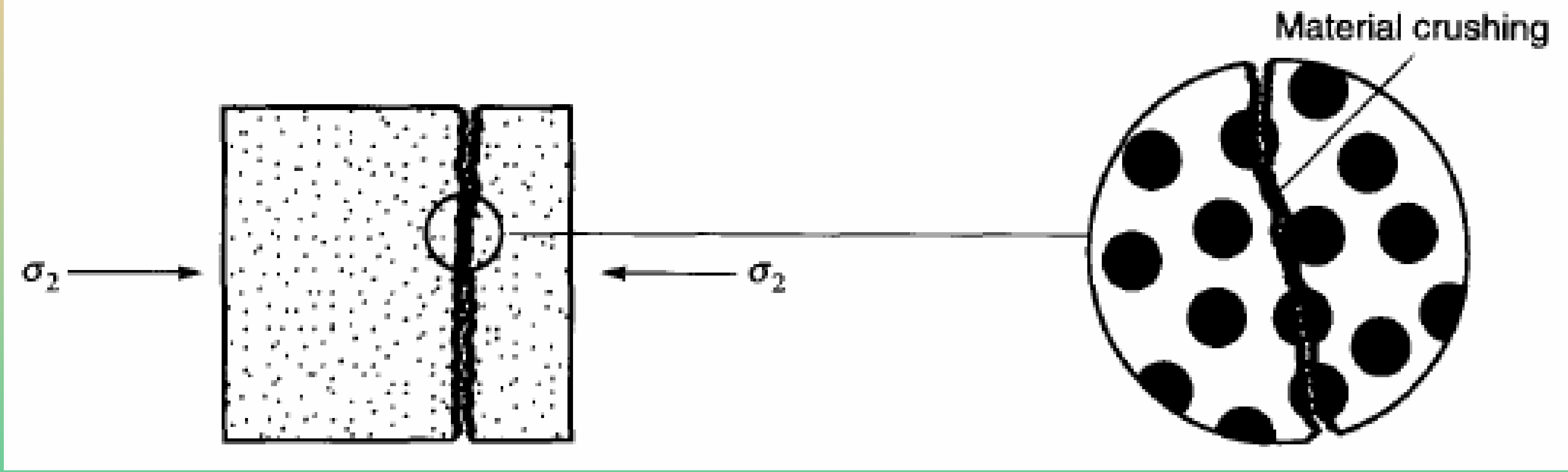
M7.3.1 Maximum Stress Criterion

Figure: Failure in compression in the 2 direction



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M7.3.1 Maximum Stress Criterion

Figure: Failure in shear in the 2-3 planes

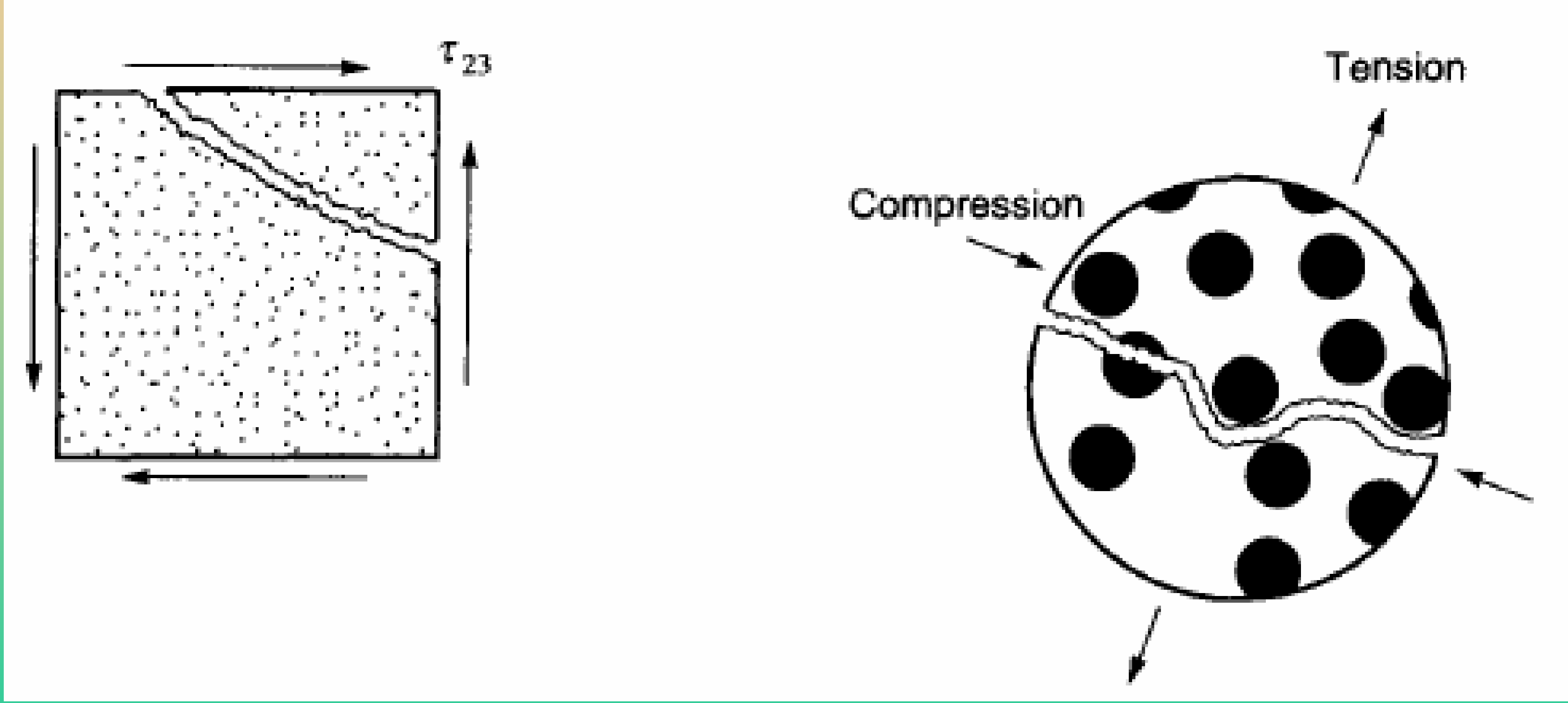

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Figure: Failure boundaries for tension failure in 1 direction in +20° layers

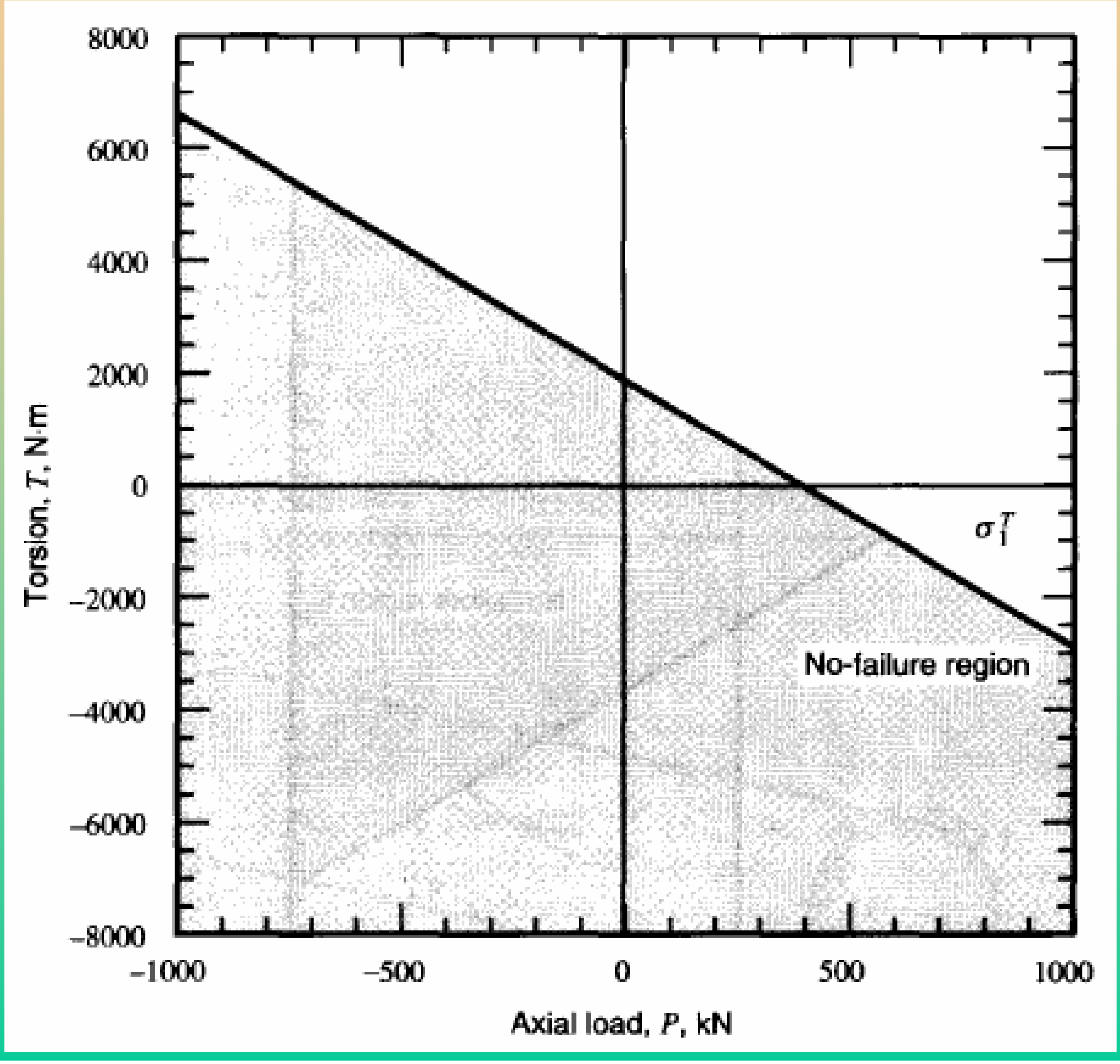

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Figure: Failure envelope for failure in 1 & 2 directions in +20° layers

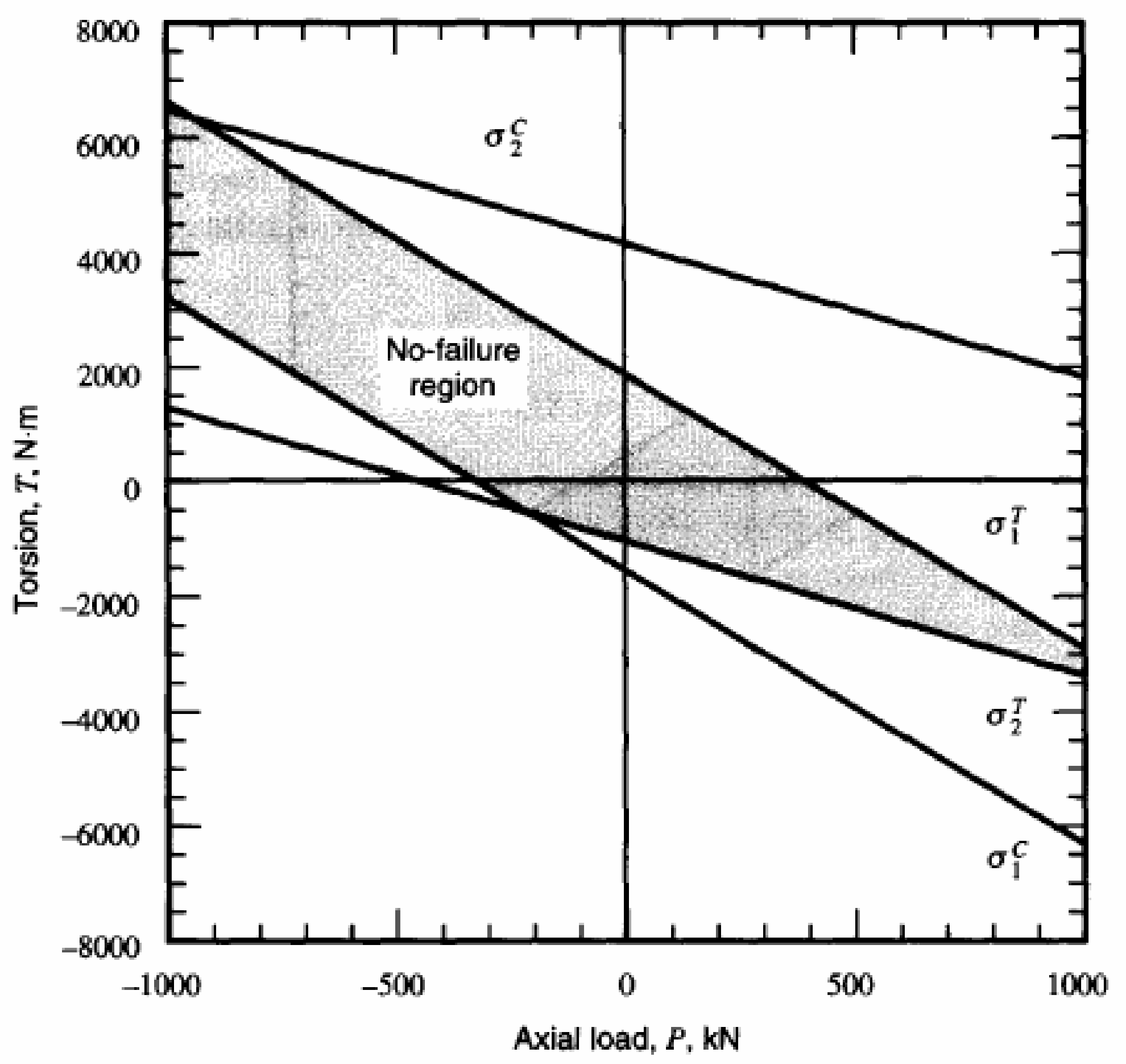

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Figure: Complete failure envelope for +20° layers.

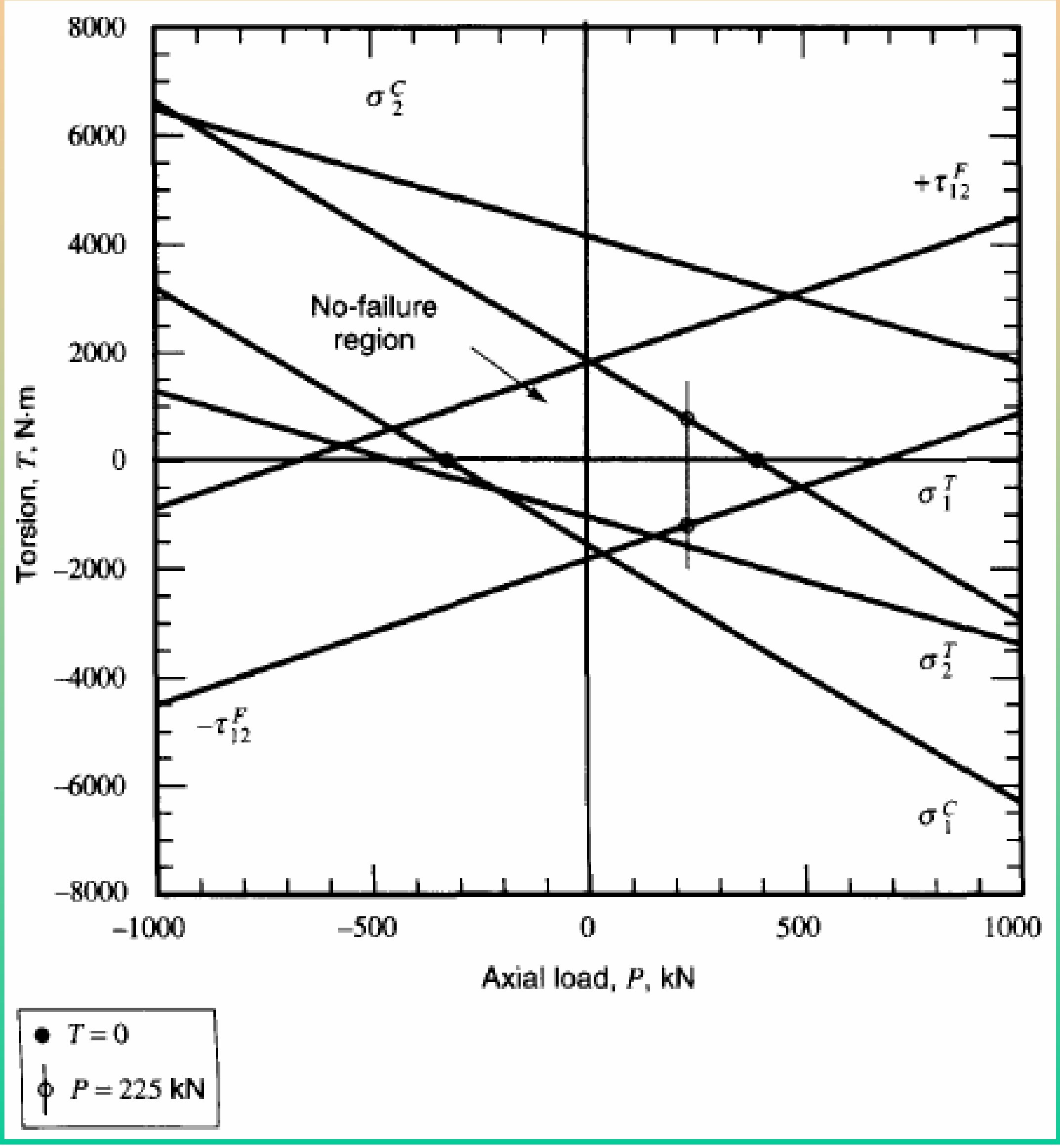

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Figure: Complete failure envelope for —20° layers.

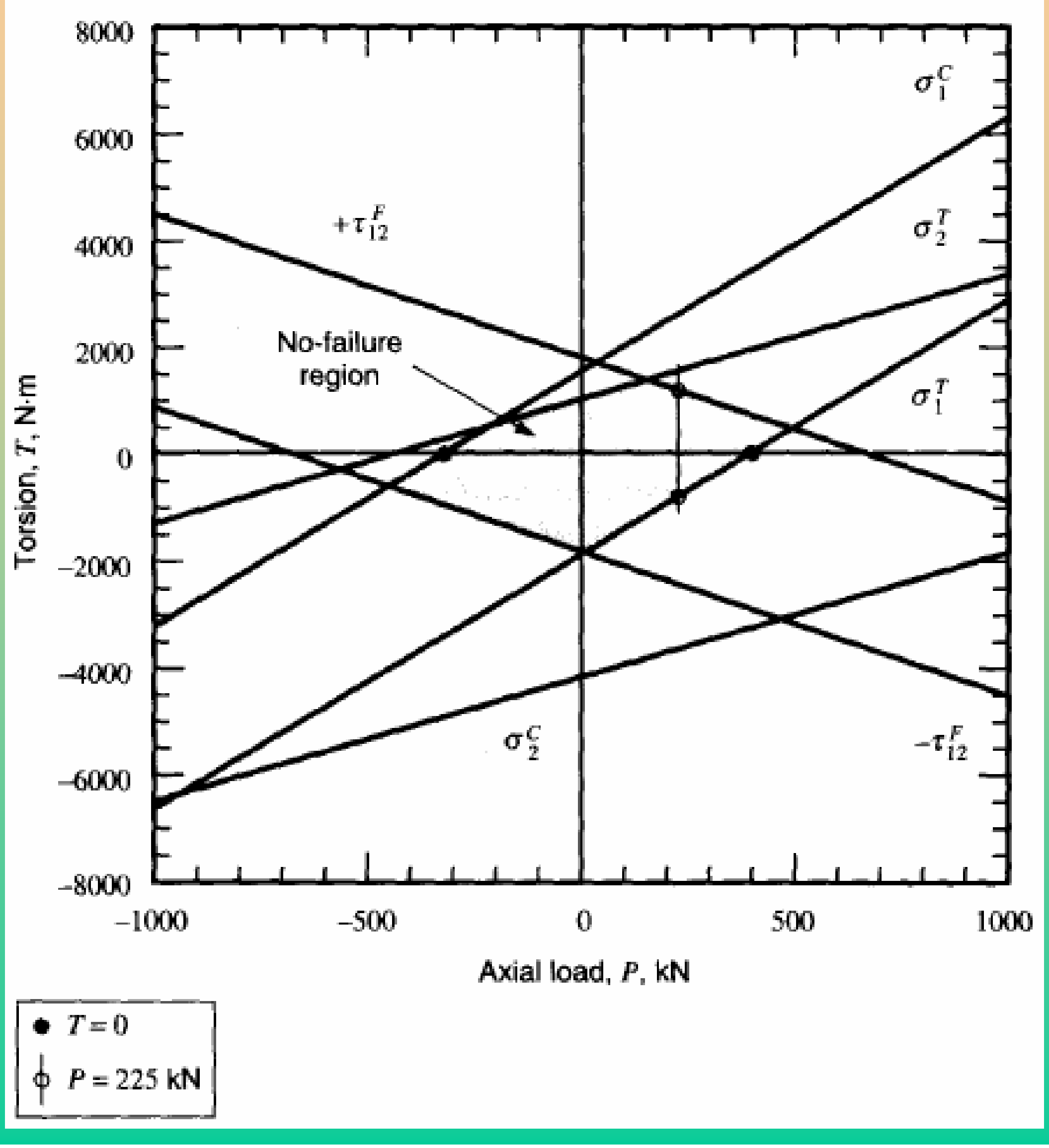

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Figure: Complete failure envelope for 0° layers.

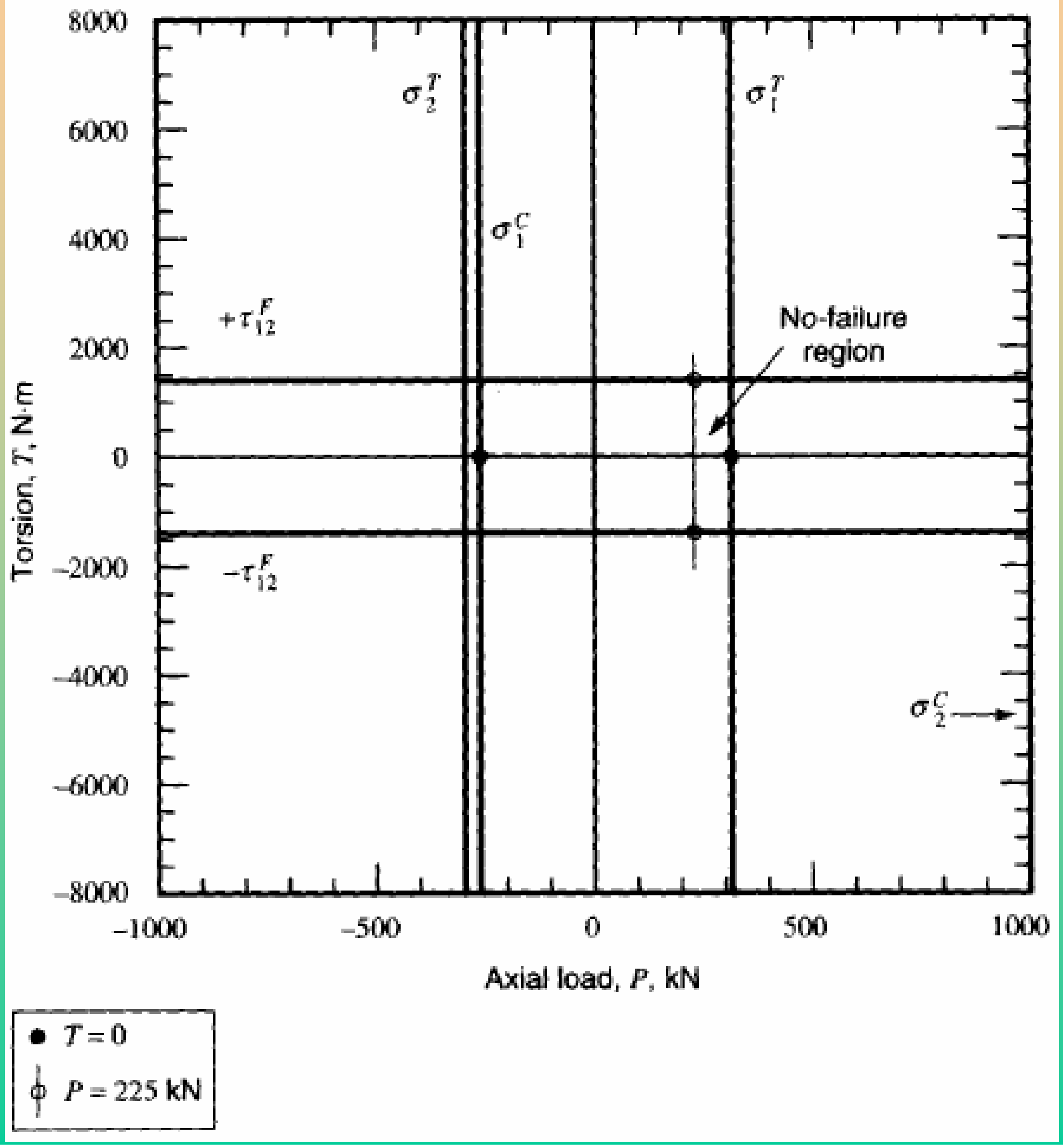

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Figure: Complete failure envelope for +20°, —20°, and 0° layers.

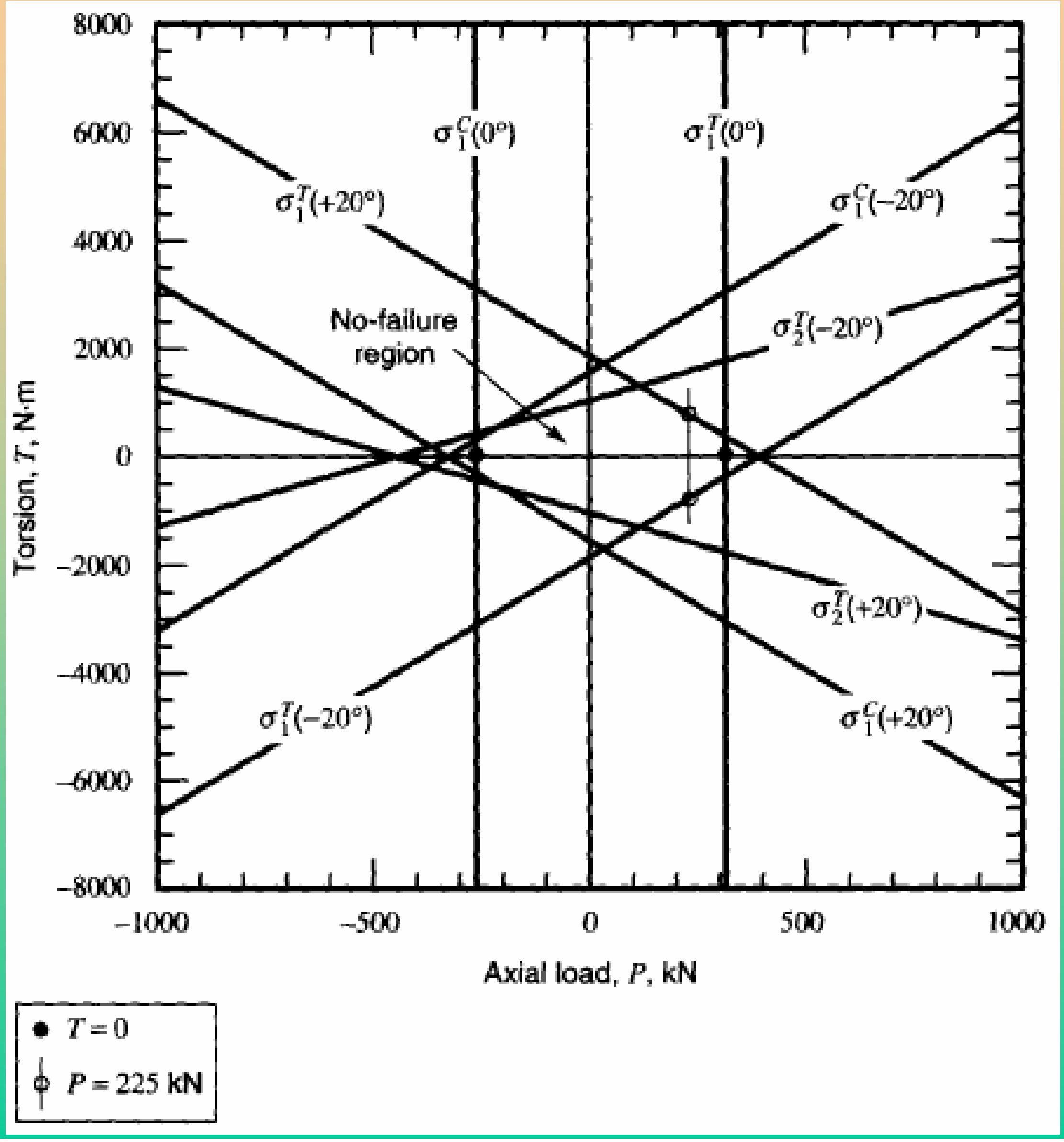

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Figure: Tsai-Wu failure ellipse for +20° layers

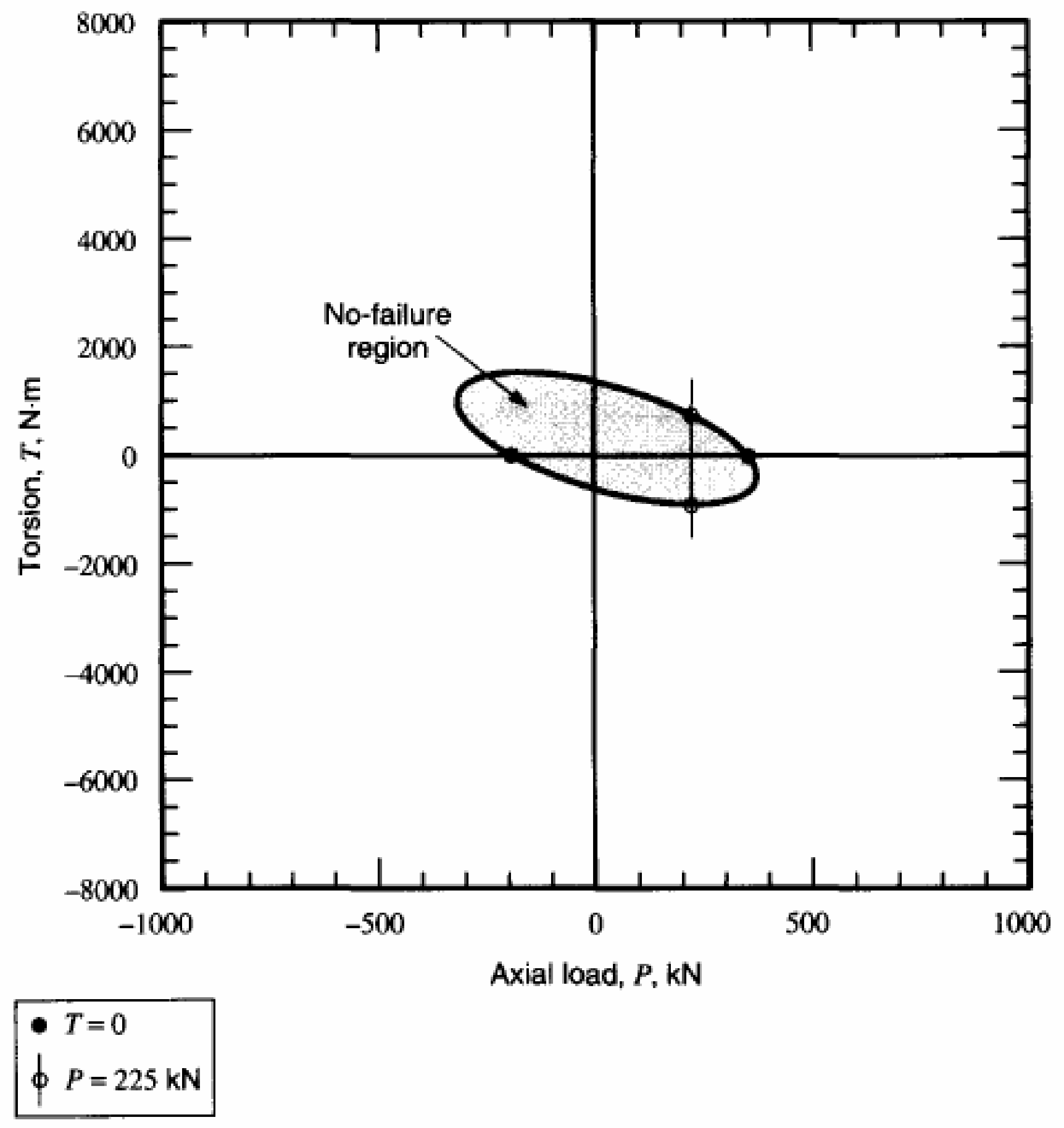

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Figure: Tsai-Wu failure ellipse for 0° layers

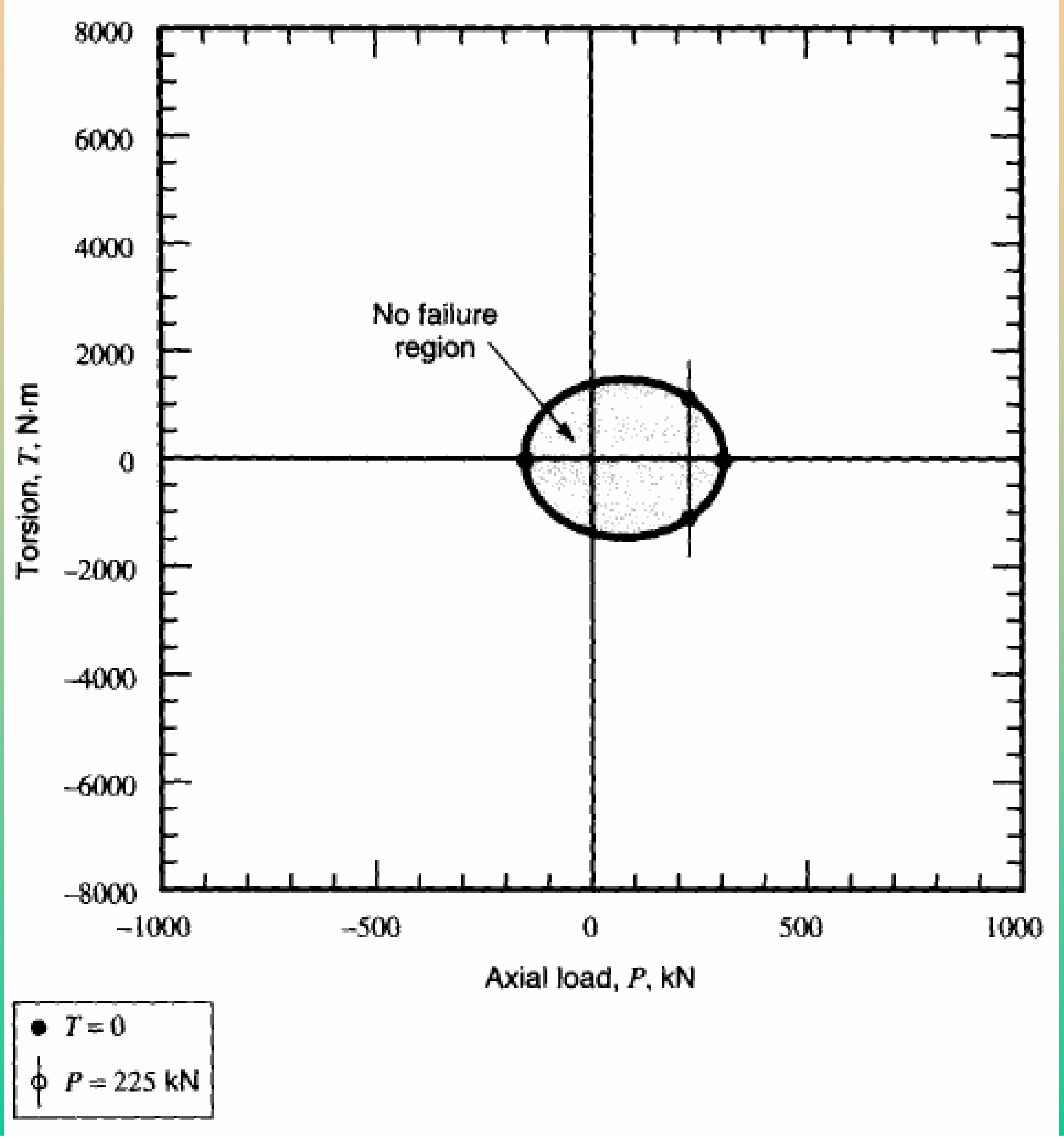

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Figure: Superposition of the Tsai-Wu failure ellipses for +20°, -20°, and 0° layers

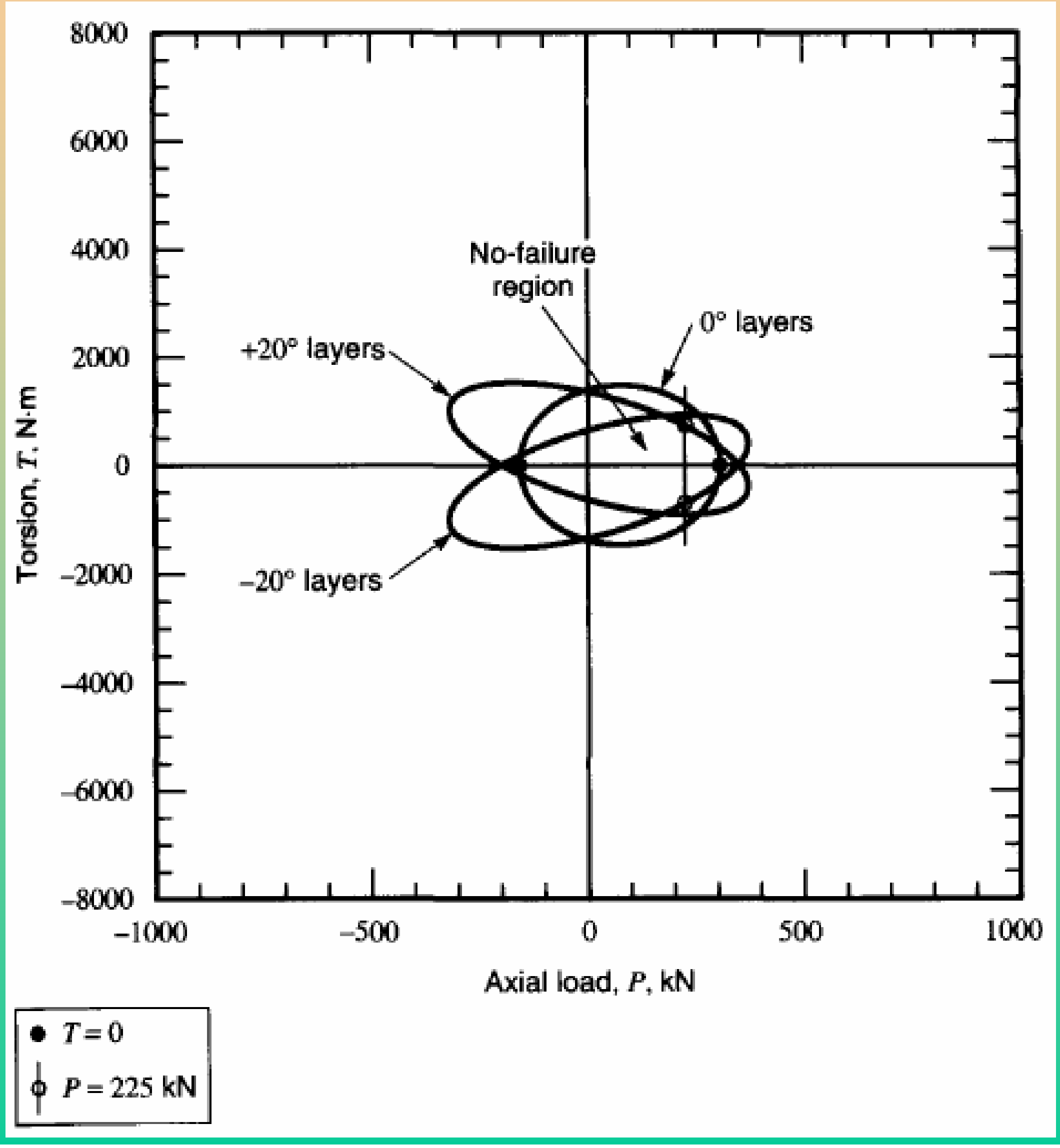

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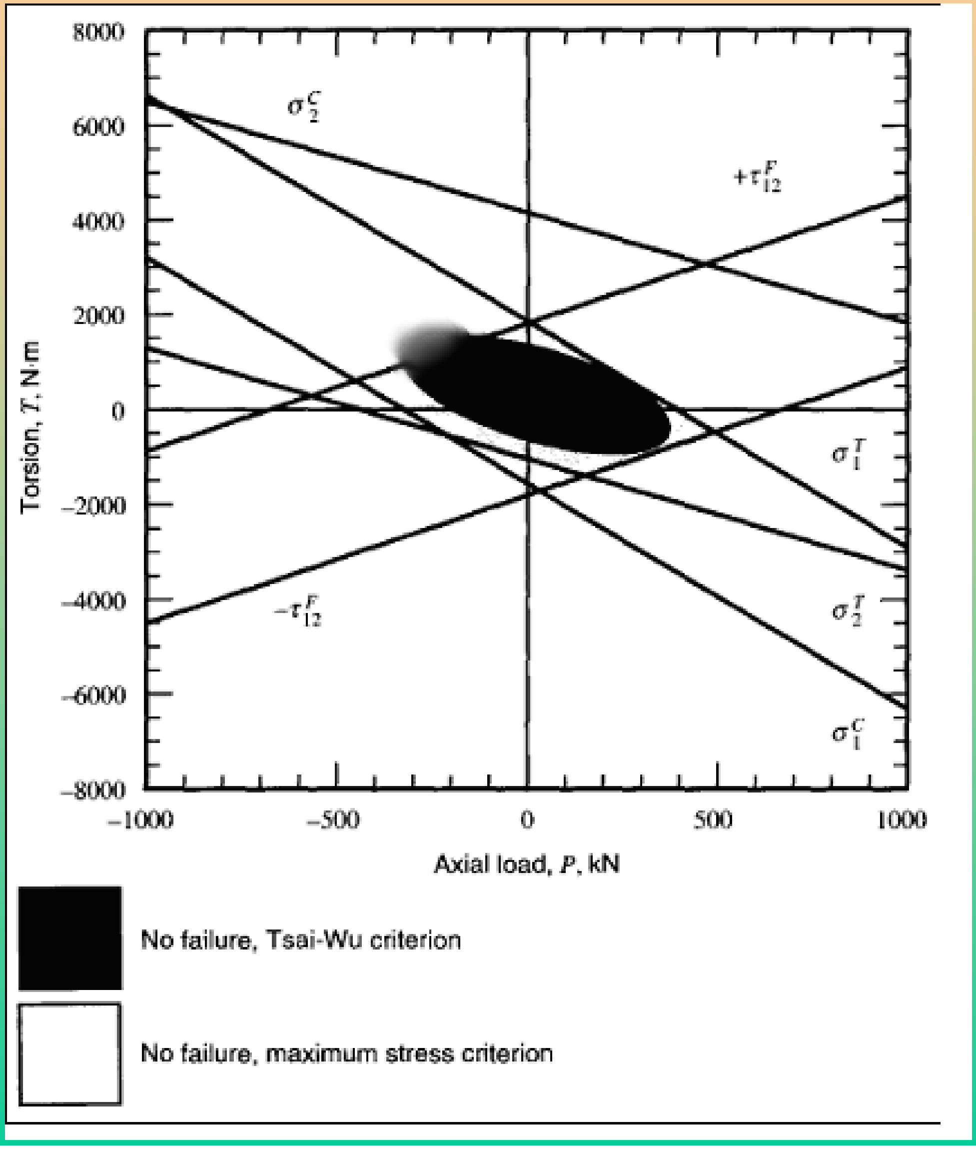
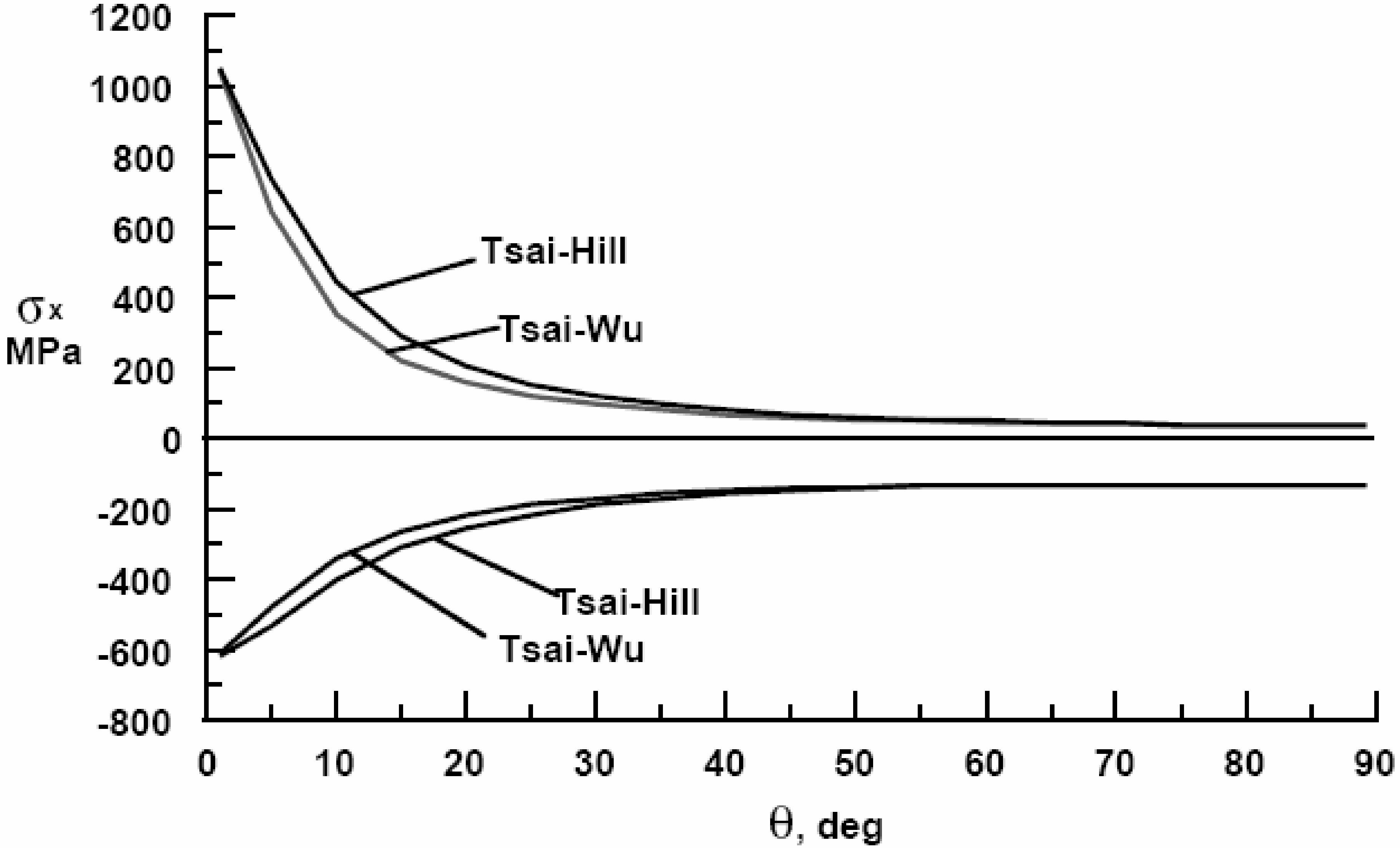
Figure: Comparison between maximum stress criterion and Tsai-Wu criterion for +20° layers for the tube subjected to combined axial load P and torsion T



Summary

Stresses must be calculated, these stresses must be used in the equations representing the failure criteria, and the results interpreted. This approach would be followed with any stress-based criterion. With a strain-based criterion, strains in the principal material system would be used rather than stresses. The key issue with any one failure criterion is, "Does it accurately predict failure for your problem?" Generally, considerable testing is required to determine if this is the case. Unfortunately, there does not appear to be one universal criterion which works well for all situations and all materials. Material properties, lamination sequence, and type of loading all seem to influence which criterion works the best. For each particular class of problems and class of materials, a careful study of test data and predictions must be conducted before generalizations can be made. We suggest that more than one criterion be used and the results compared, as we have done here. Competing views are helpful!

Comparison of Failure Theories



Comparison of Failure Theories

Theory	Physical basis	Operational convenience	Required operational convenience
Maximum stress	Tensile behaviour of brittle material	Inconvenient	Few parameters by simple testing
Maximum strain	Tensile behaviour of brittle material Some stress interaction	Inconvenient	Few parameters by simple testing

Continued.....

Theory	Physical basis	Operational convenience	Required operational convenience
Deviatoric strain energy (Tsai-Hill)	Ductile behavior of anisotropic materials "Curve fitting" for heterogeneous brittle composites	Can be programmed Different functions required for tensile and compressive strengths	Biaxial testing is needed in addition to uniaxial testing
Interactive tensor polynomial	Mathematically consistent Reliable "curve fitting"	General and comprehensive; operationally simple	Numerous parameters Comprehensive experimental program needed