Module 5

M5.Laminated Composites - I

Learning Units of Module 5

M5.1 Introduction to Mechanics of Plates

M5.2 Macromechanics of Laminate

M5.3 Stress-Resultants in Laminate

Learning Unit M5.1

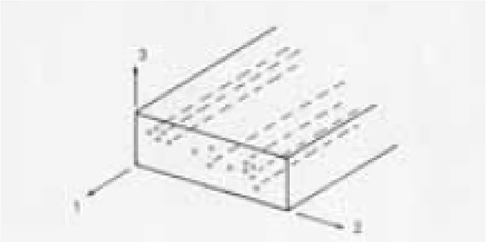
M5.1 Introduction of Mechanics of Plate

Laminated Plates

Classical lamination theory makes many assumptions. Higher order theories exist.

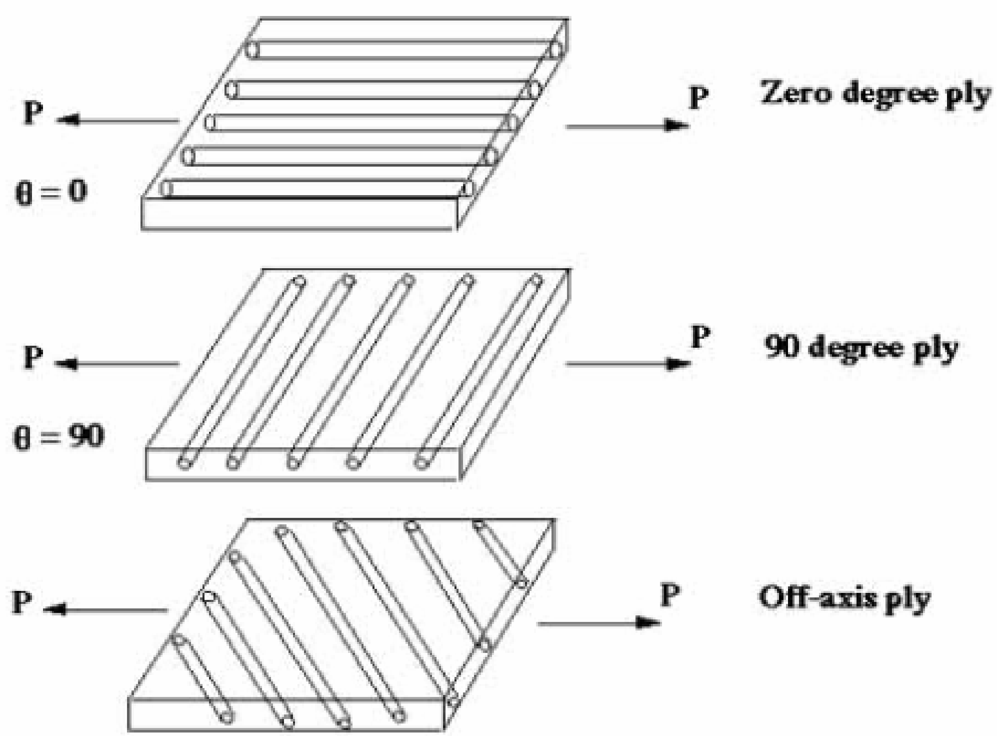
Laminate Theory Introduction

- A "Ply" is a composite layer in which the fibers are typically arranged in one direction (structural building block)
- Ply-by-ply analysis used to determine/understand the stress state in the composite material
- Composite behavior (on a macro scale) can be predicted by summing the contributions of the individual plies
- For unidirectional fiber reinforced materials, transversely isotropic behavior is typically assumed in the laminae (same elastic properties in the 2 and 3 material directions

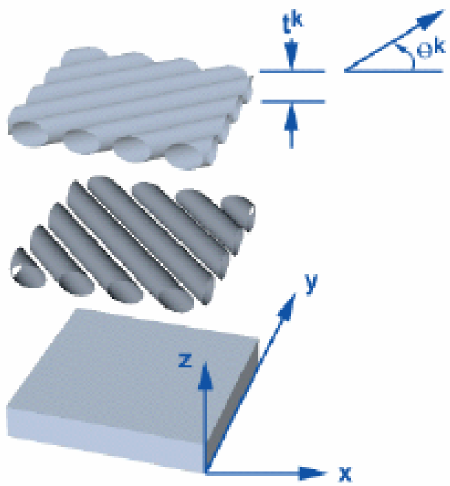


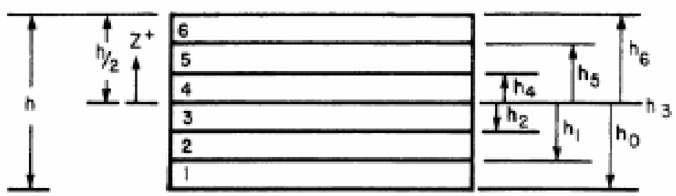
- Orthotropic materials have longitudinal (1) properties that are very different from the transverse (2 and 3) properties

Laminate Theory Introduction



Laminate Theory Introduction





Layer Nomenclature

- In the principal material coordinate system the stiffness relation for the kth ply is given by

$$\sigma_i^k = Q_{ij}^k \left(\varepsilon_j^k - \alpha_j^k \Delta T^k\right) \; (i, j = 1, 2, 6)$$

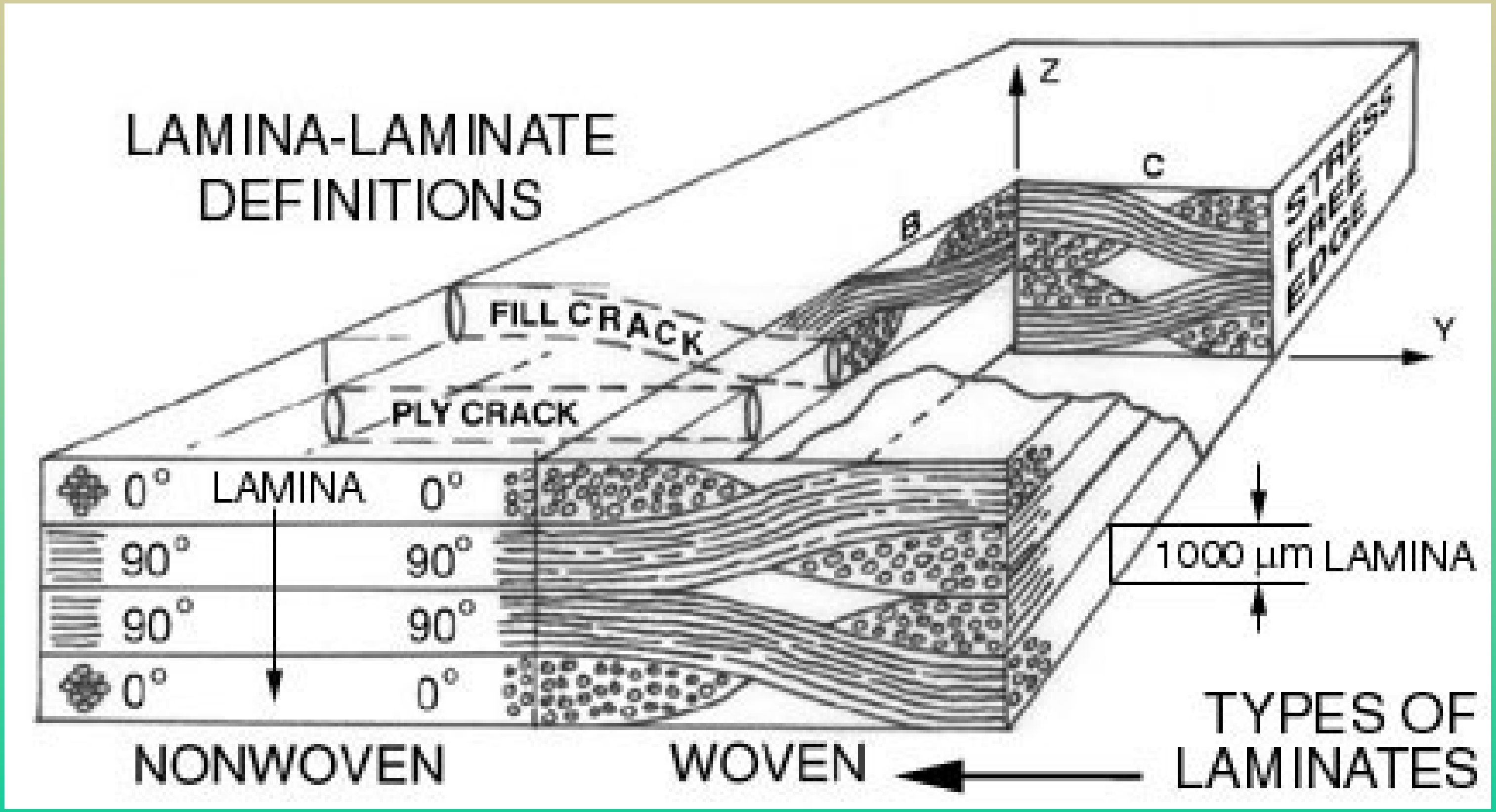
- The stiffness coefficients expressed in terms of the lamina engineering material properties are defined below:

$$Q_{11}^k = \frac{E_1^k}{1 - \nu_{12}^k \, \nu_{21}^k} \qquad Q_{12}^k = Q_{21}^k = \frac{\nu_{21}^k E_1^k}{1 - \nu_{12}^k \, \nu_{21}^k}$$

$$Q_{22}^k = \frac{E_2^k}{1 - \nu_{12}^k \, \nu_{21}^k} \qquad Q_{66}^k = G_{12}^k$$

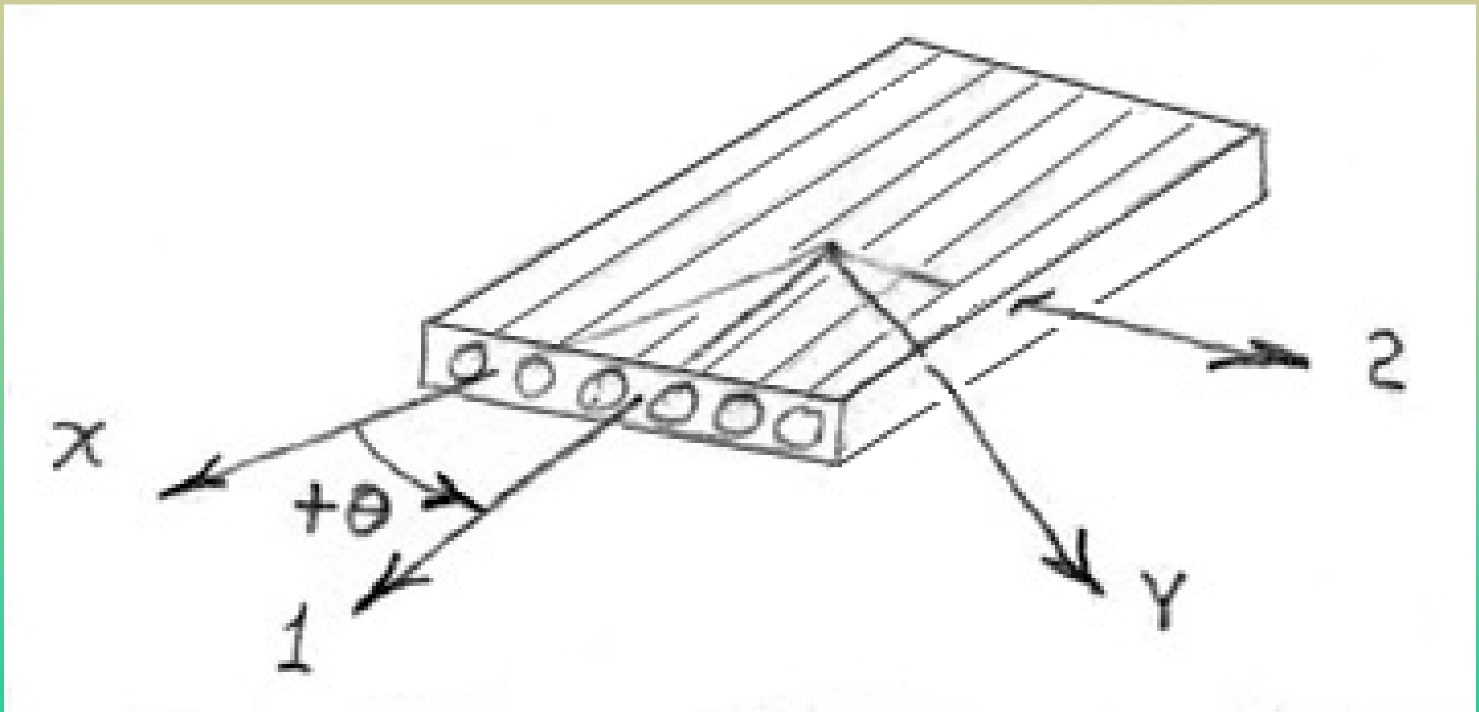
Lamina & Laminate Definition

- Figure shows the construction of a laminated plate with individual lamina layers (Lamina-Laminate Definitions). In practice laminated plates consist of hundreds stacked lamina but here we study simple laminated structures.



Analytic Model:

- The figure shows an orthorhombic lamina in plane stress with the 1-2 axes be aligned with the fiber axis.



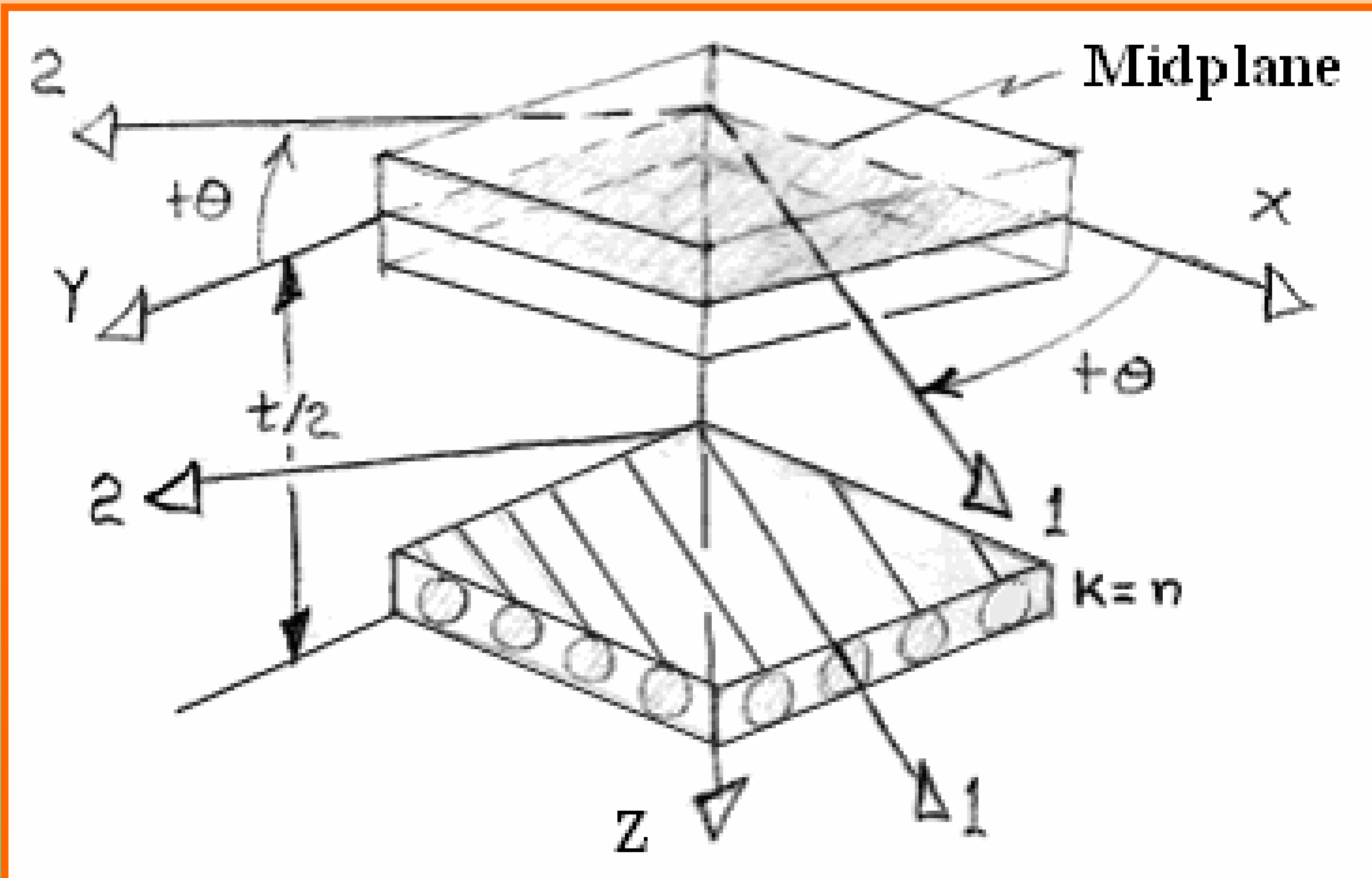


Figure: Lamina-Laminate Configuration Geometry Defined

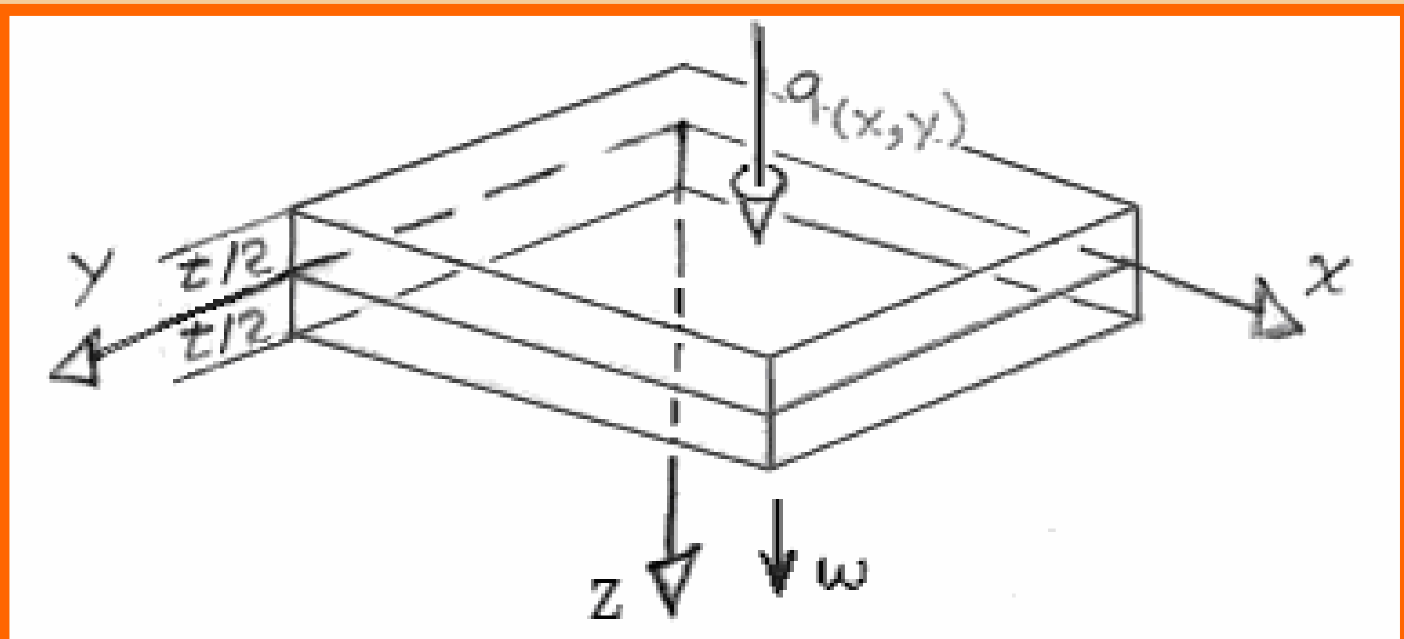
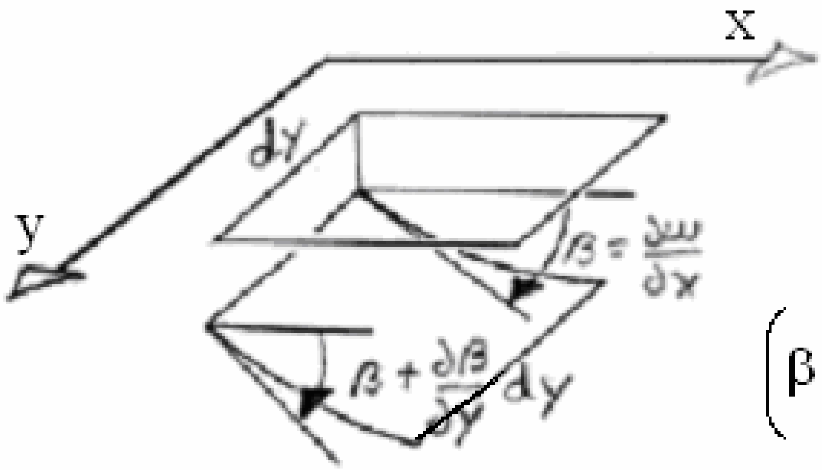


Figure: Thin Plate Deflections



$$\text{Twist} = K_{xy}$$

$$\left(\beta + \frac{\partial \beta}{\partial y} dy\right) - \beta = \frac{\partial \beta}{\partial y} dy$$

$$\therefore \text{Twist} = K_{xy} = \frac{\text{Angle change}}{\text{Angle length}} = \frac{\frac{\partial \beta}{\partial y} dy}{dy} = -\frac{\partial^2 w}{\partial x \, \partial y}$$

Figure: Exaggerated Thin Plate Deflections and Twists Defined

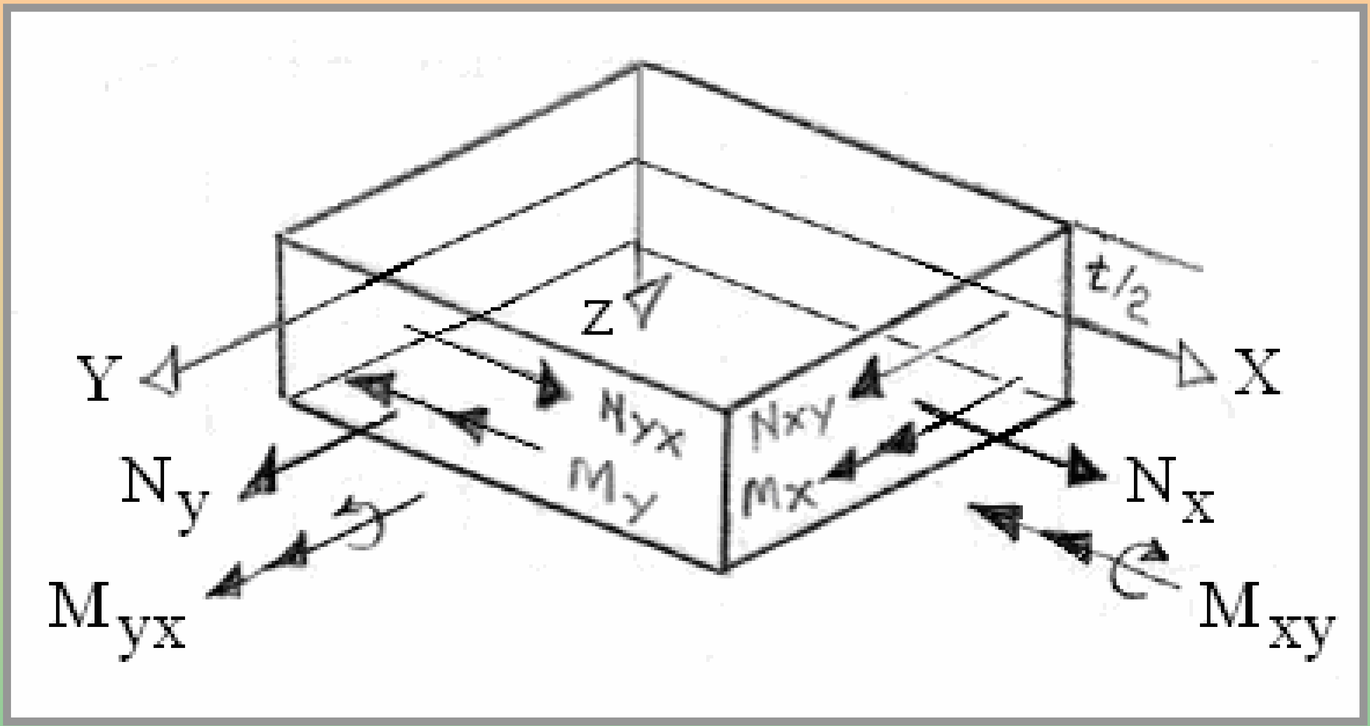


Figure: Laminate Plate In-Plane Loads and Moments Defined

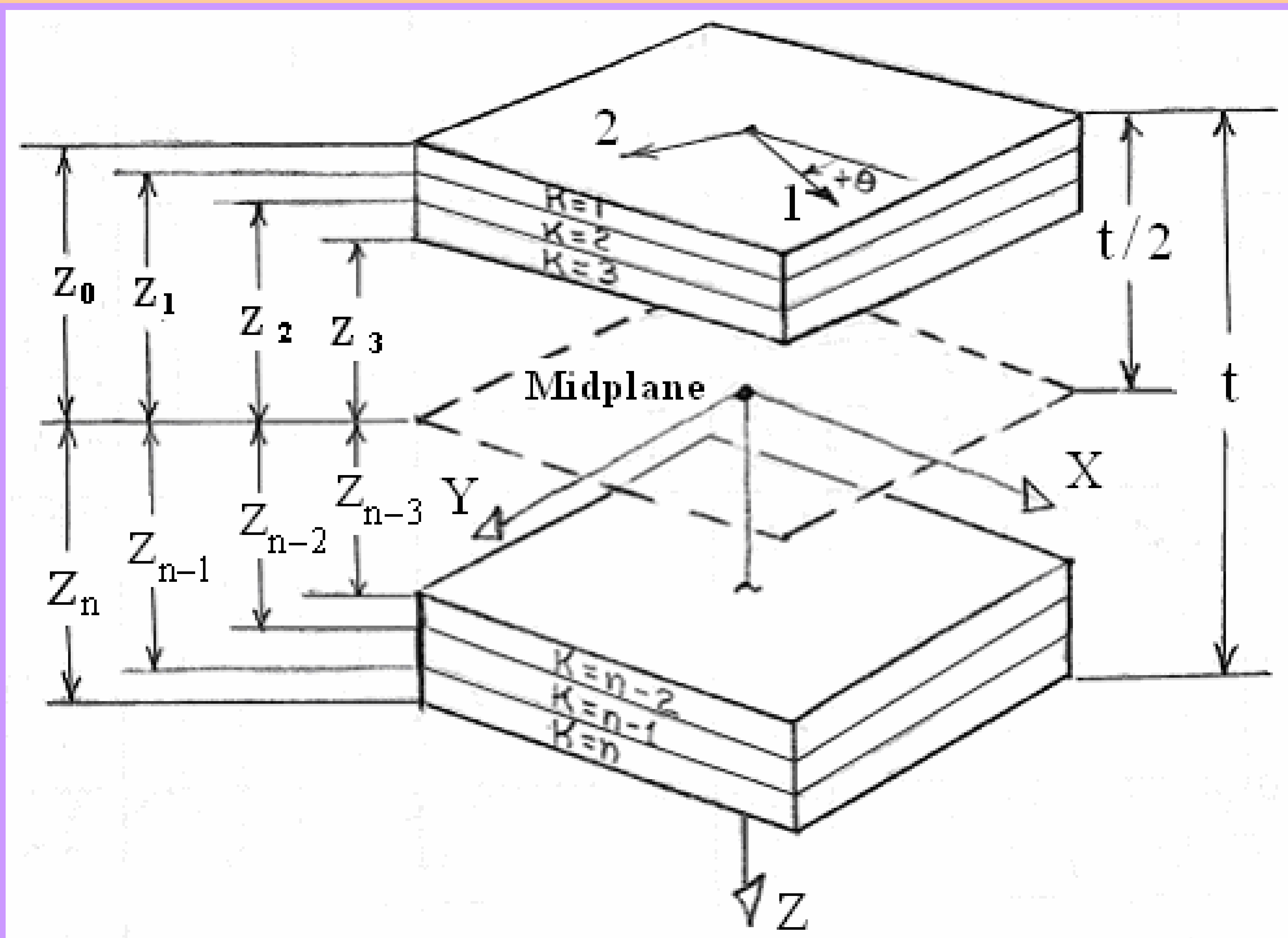


Figure: Lamina Positions Defined in the Laminate for Integration of Resultant Loads

Geometry & Deformation

- Figure shows the construction of a laminated plate with individual lamina layers (Lamina-Laminate Definitions). In practice laminated plates consist of hundreds stacked lamina but here we study simple laminated structures.

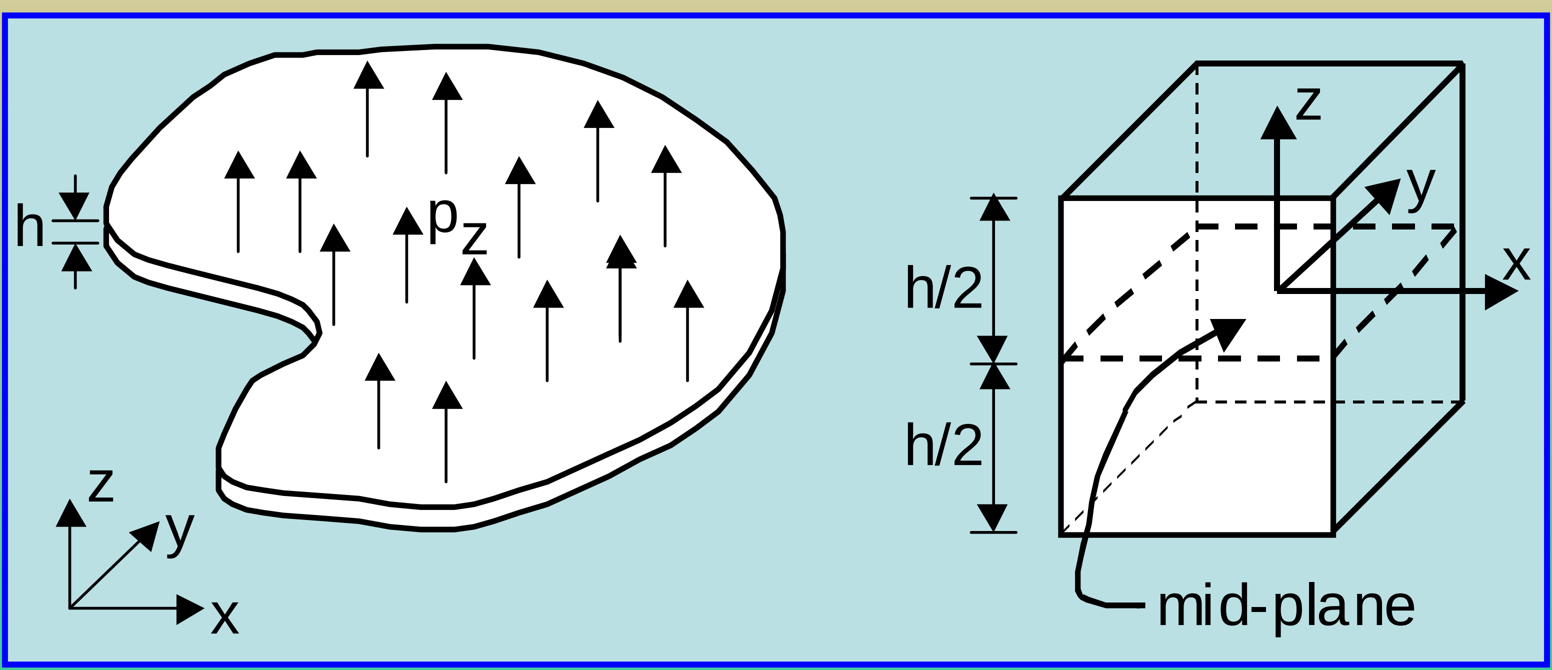
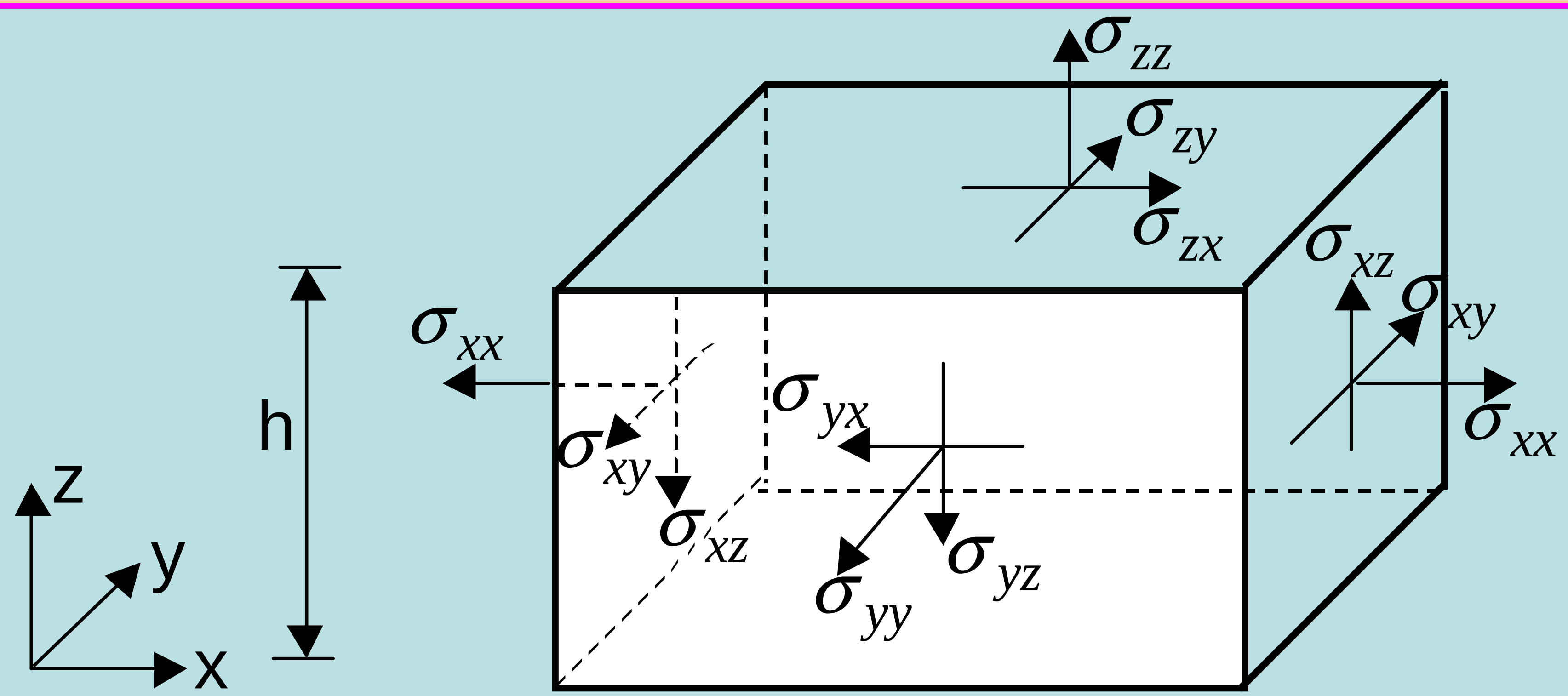


Figure: Plate Geometry

Continued……



note: stresses not shown all faces

Figure: Free Body of Stress Components

Continued……

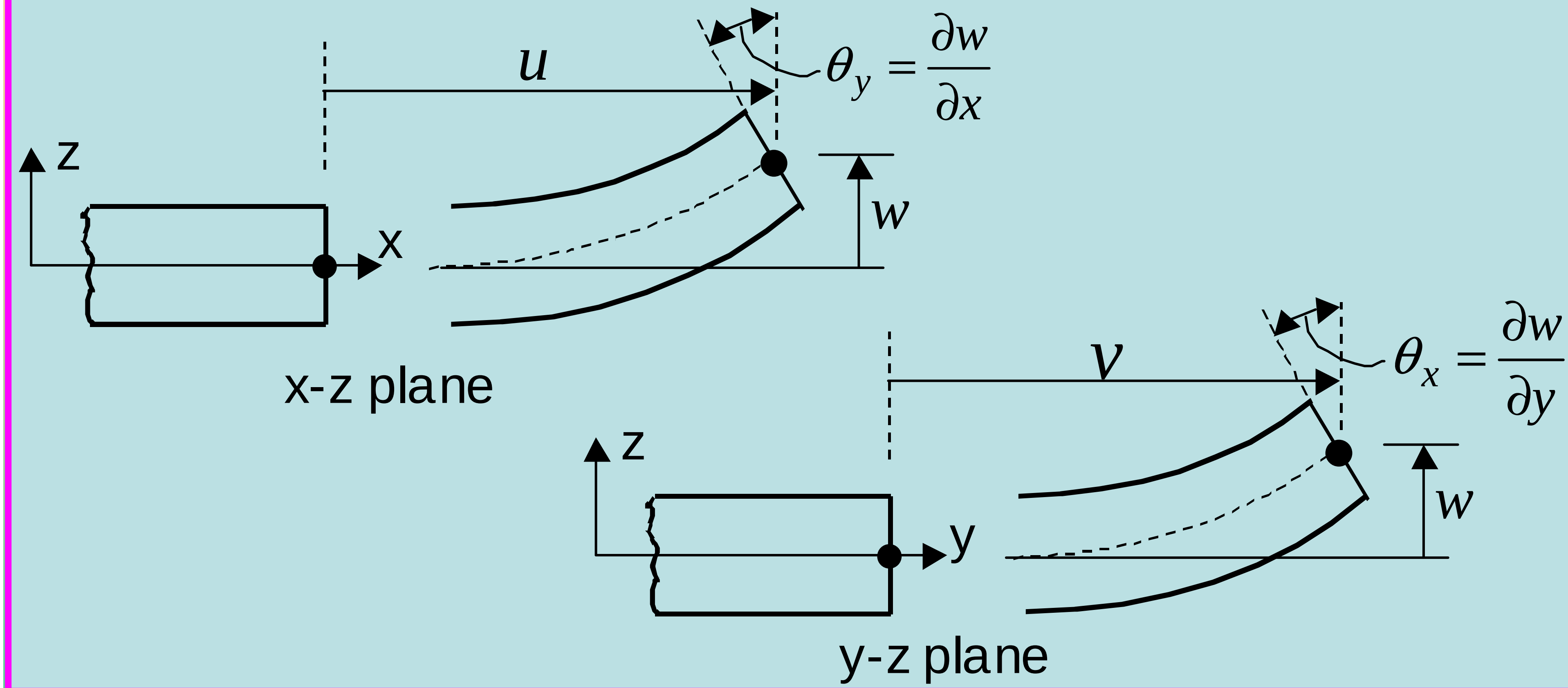


Figure: Mid-plane Displacements

Continued……

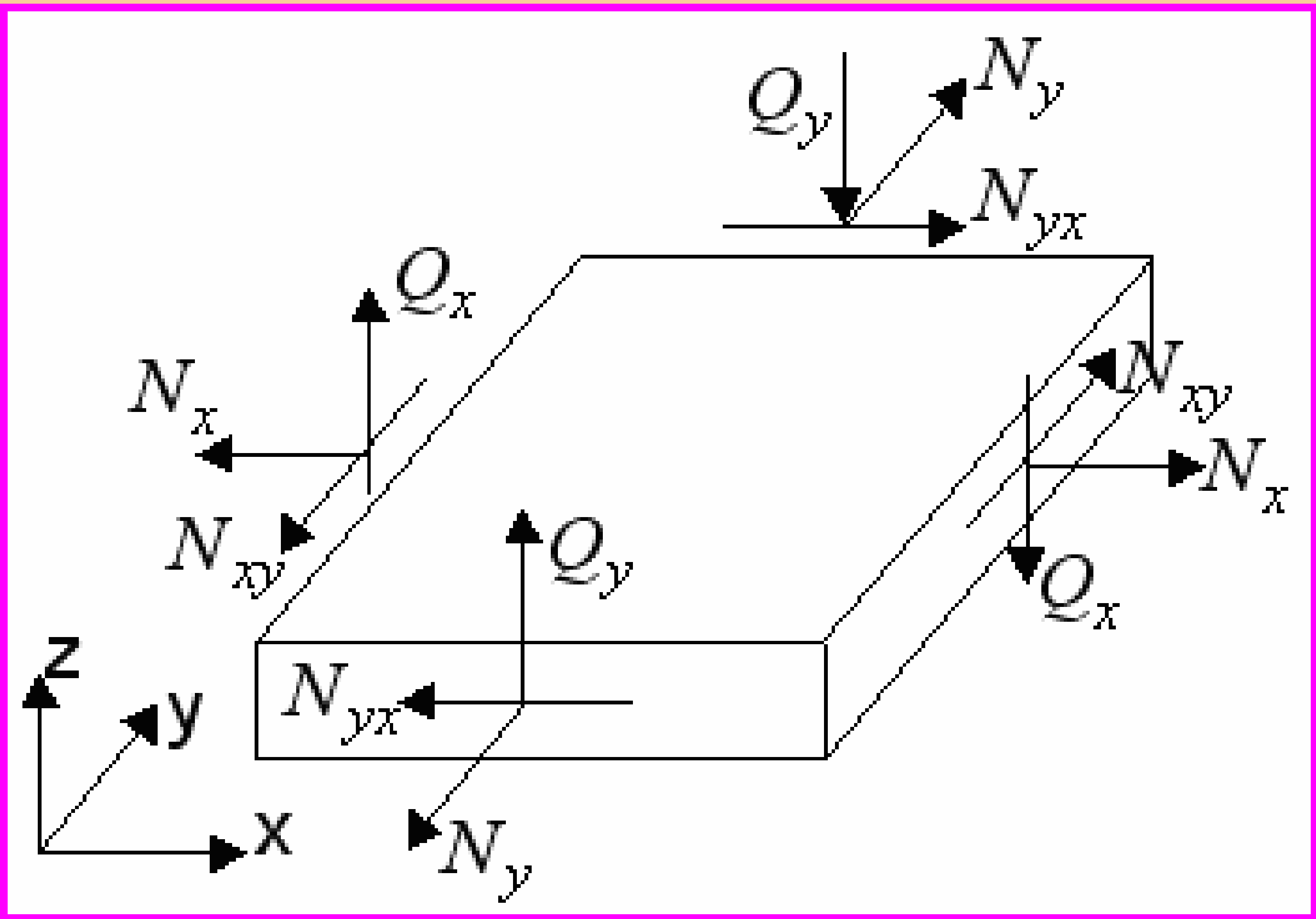


Figure: Force Stress Results

Continued……

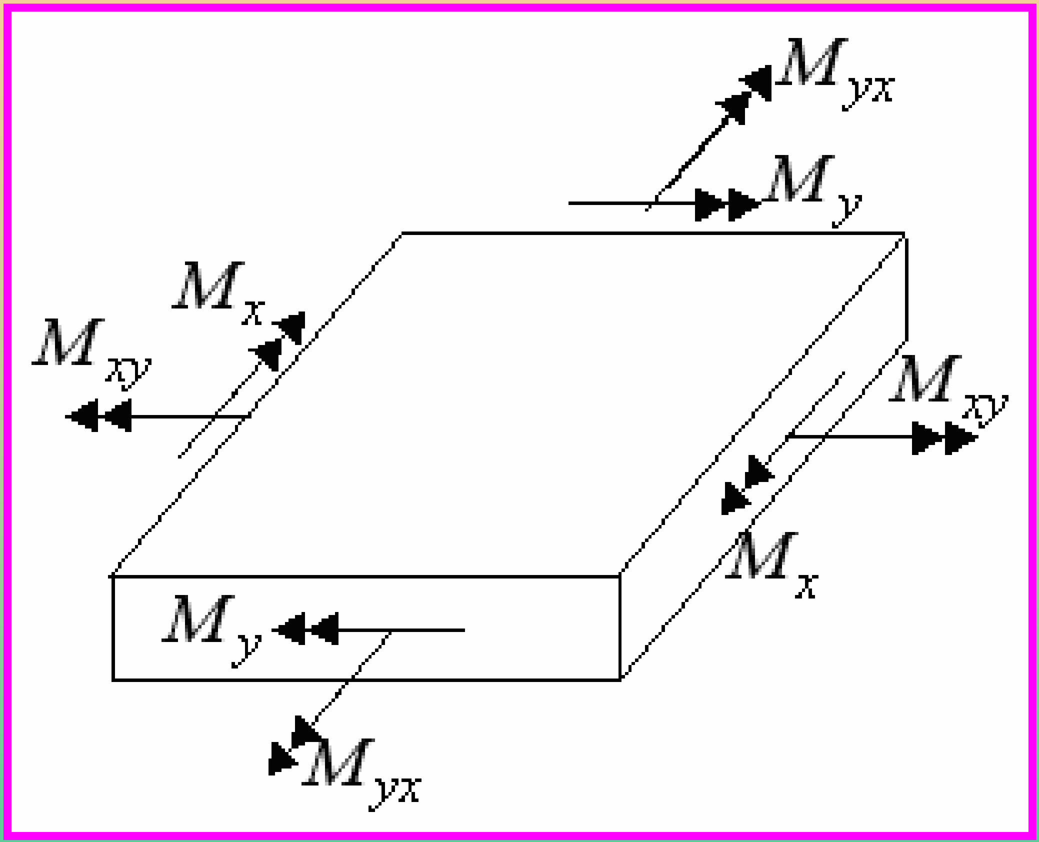
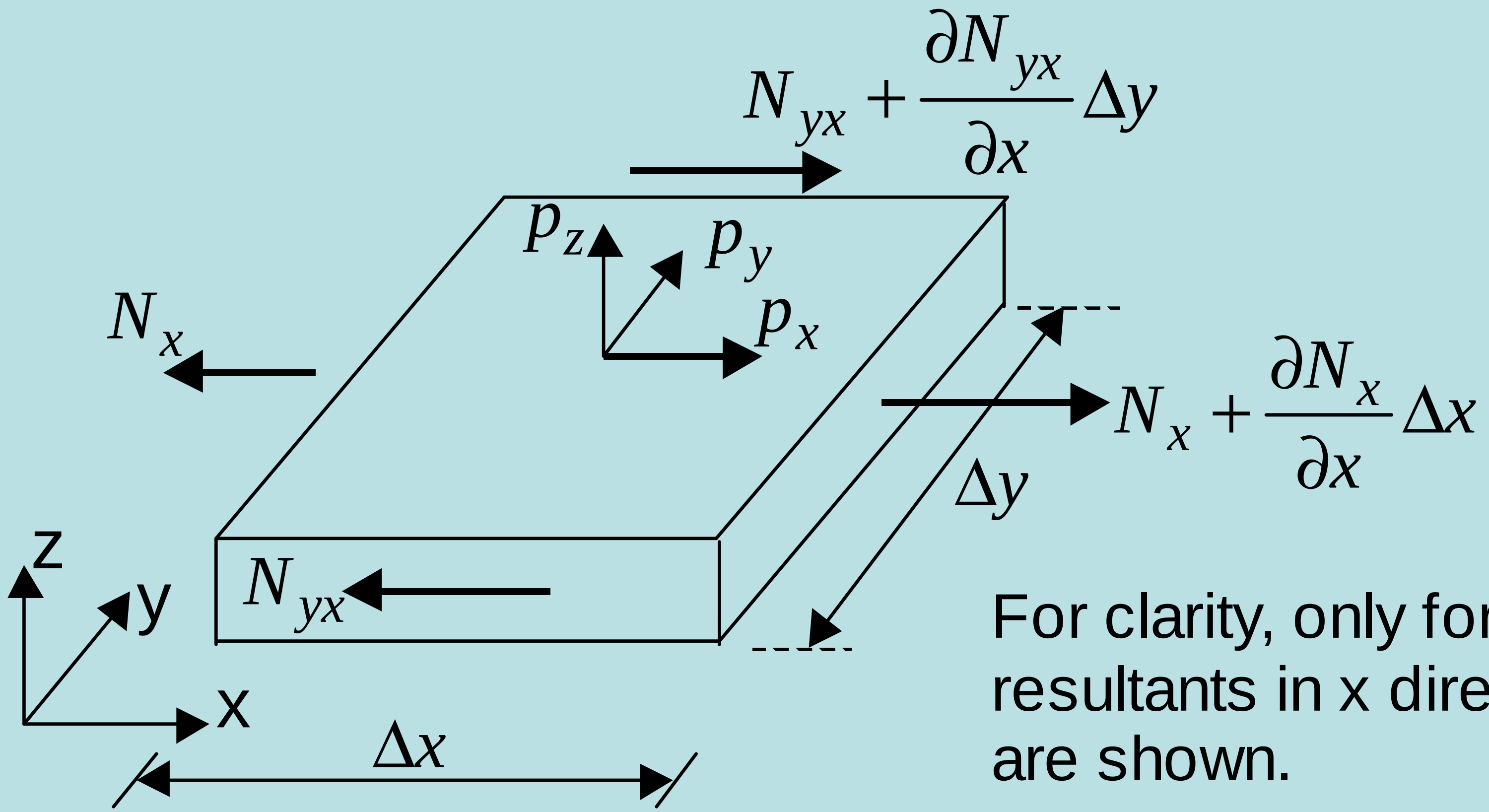


Figure: Moment Stress Results

Equilibriums in terms of Stress Resultants



For clarity, only force resultants in x direction are shown.

Figure: Equilibrium in X-direction

Learning Unit M5.2

M5.2 Macromechanics of Laminate

Strain-Displacement relations

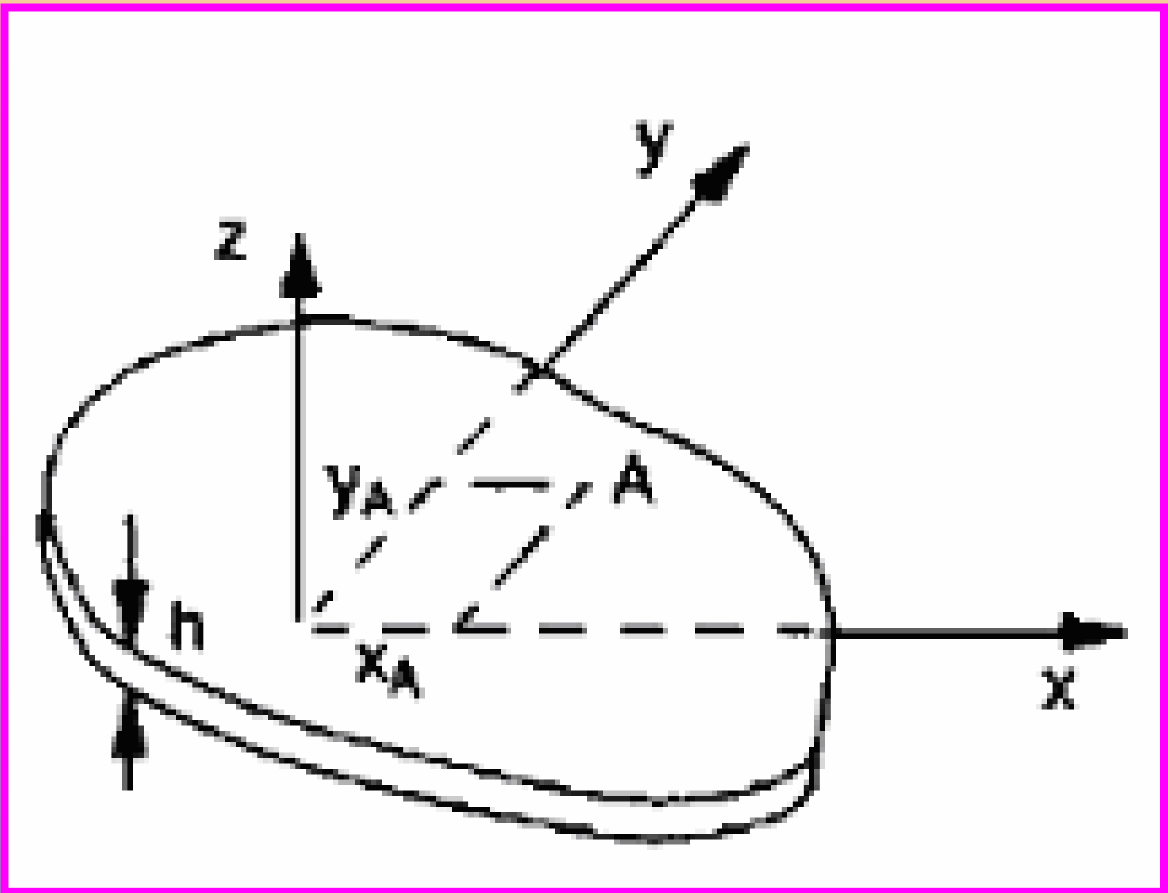


Figure M5.2.2.1 (a) Plate in which the xy-plane coincides with the mid-plane of the plate

Strain-Displacement relations

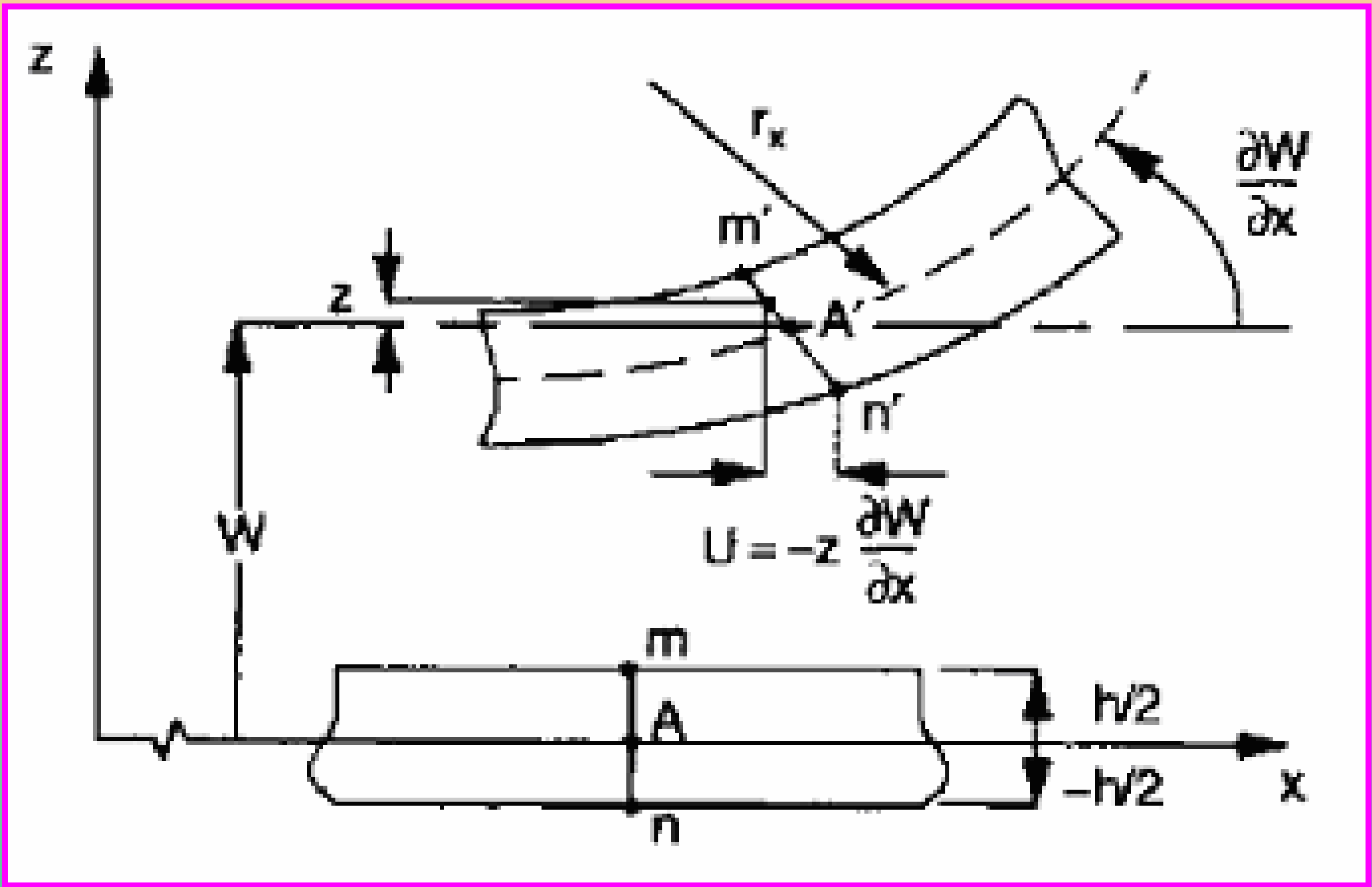


Figure M5.2.2.1 (b) Plate geometry for classical lamination theory

Stress-Strain Relations

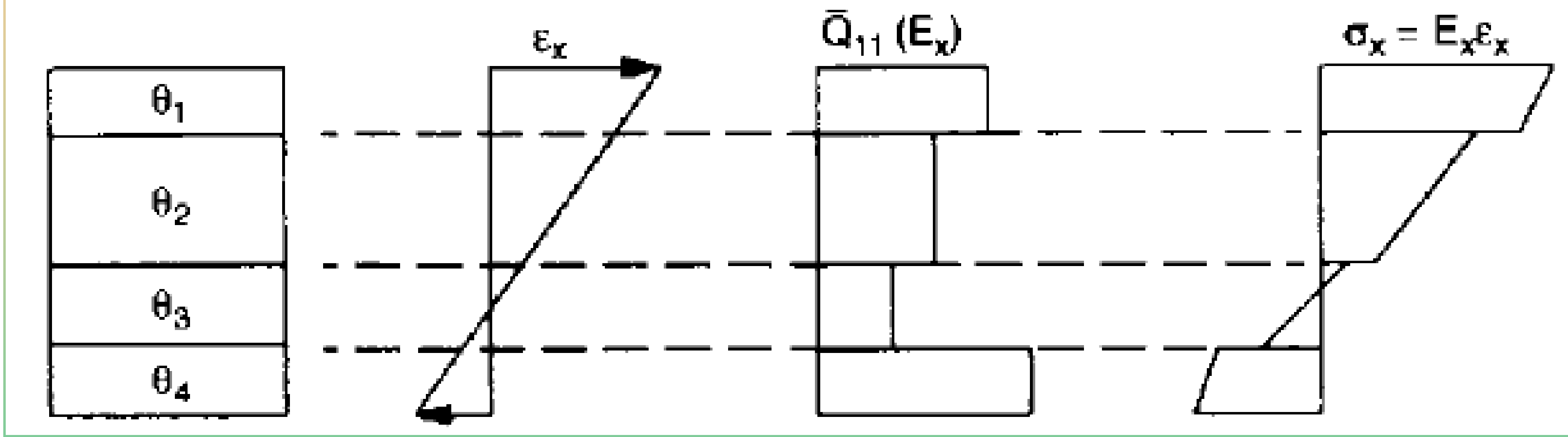


Figure: Stress variations in a variable-modulus materials

Laminate Load-Strain and Moment- Curvature Relations

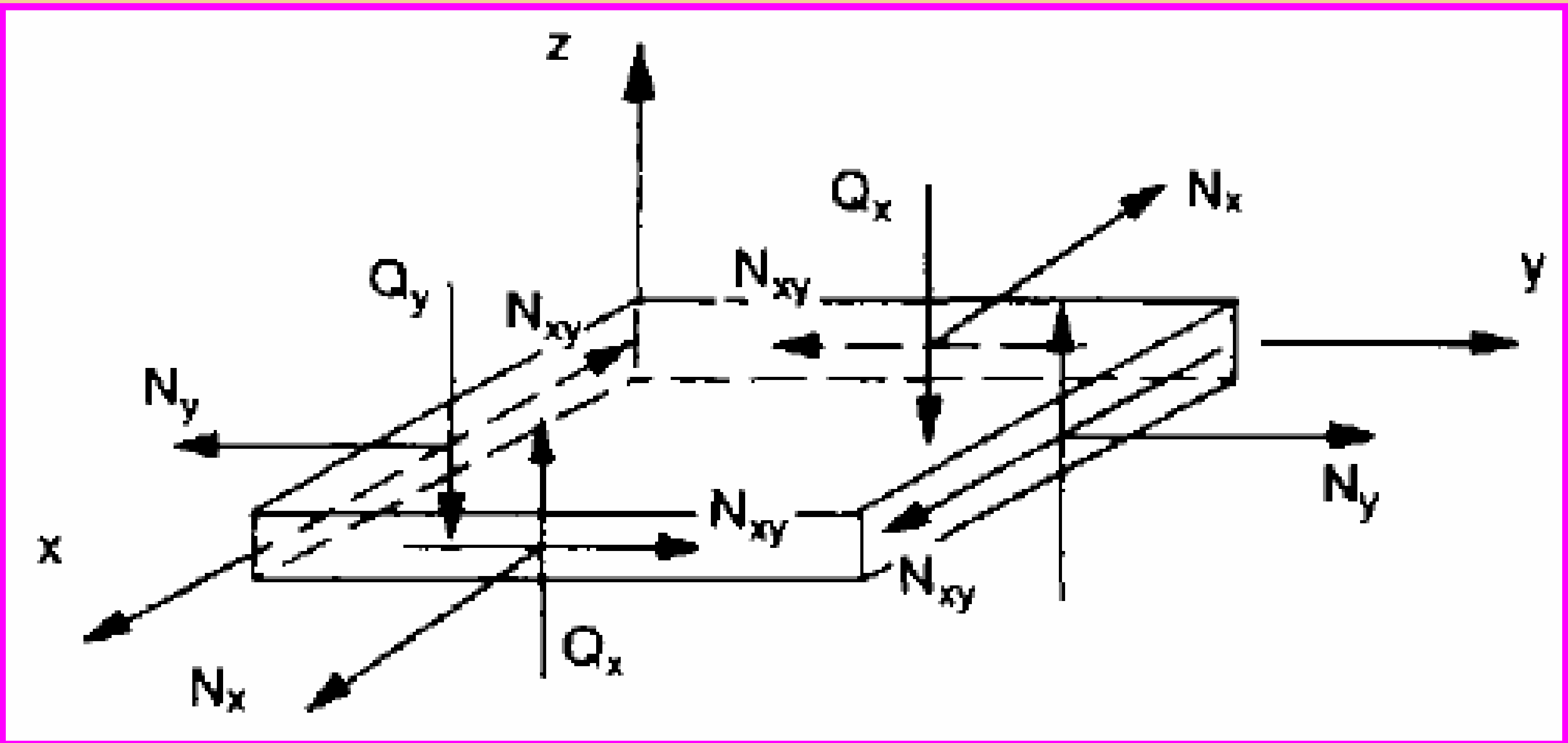


Figure: Positive sign convention for laminate loads

Laminate Load-Strain and Moment- Curvature Relations

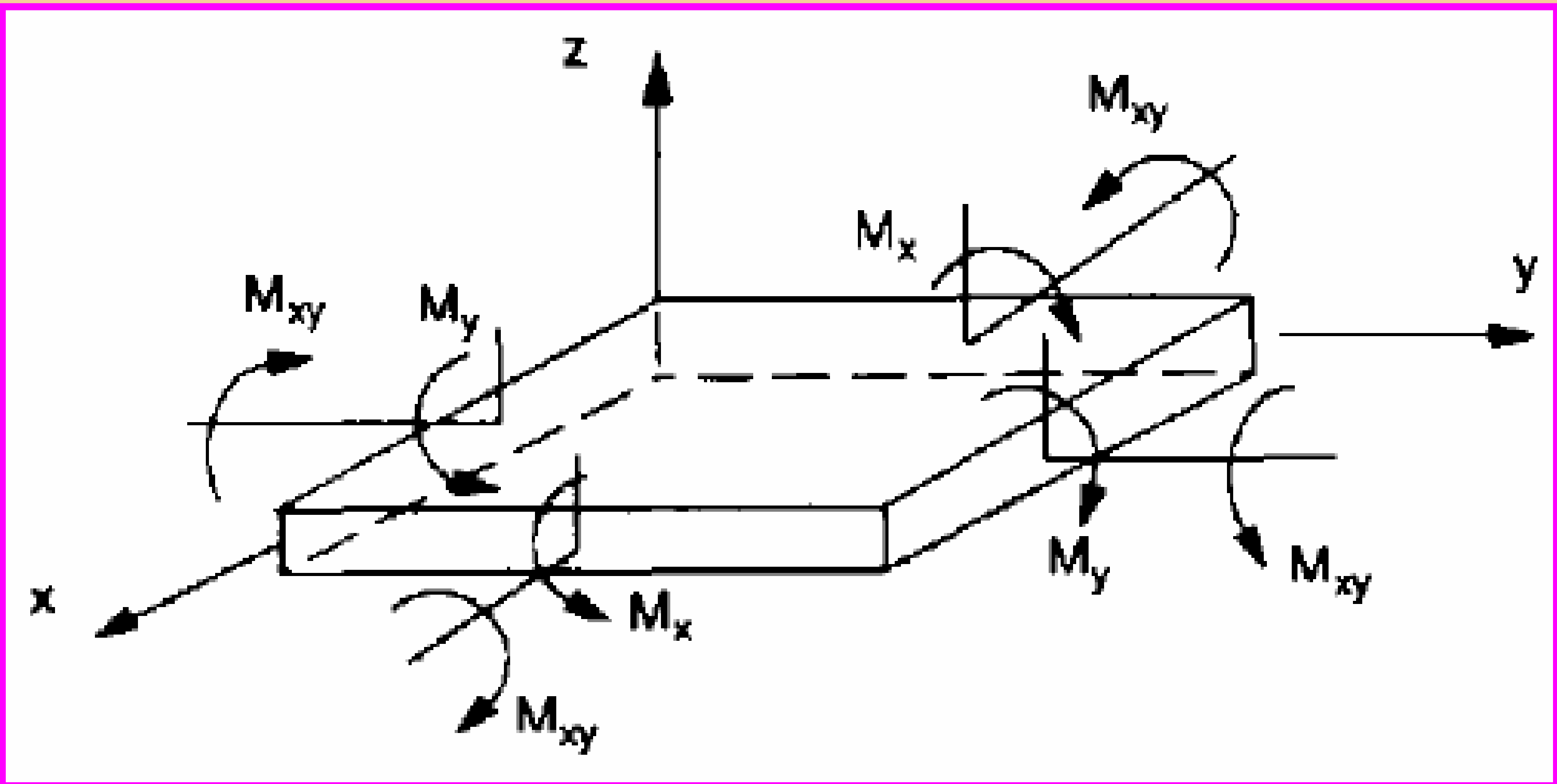


Figure: Positive sign convention for laminate moments

Laminate Load-Strain and Moment- Curvature Relations

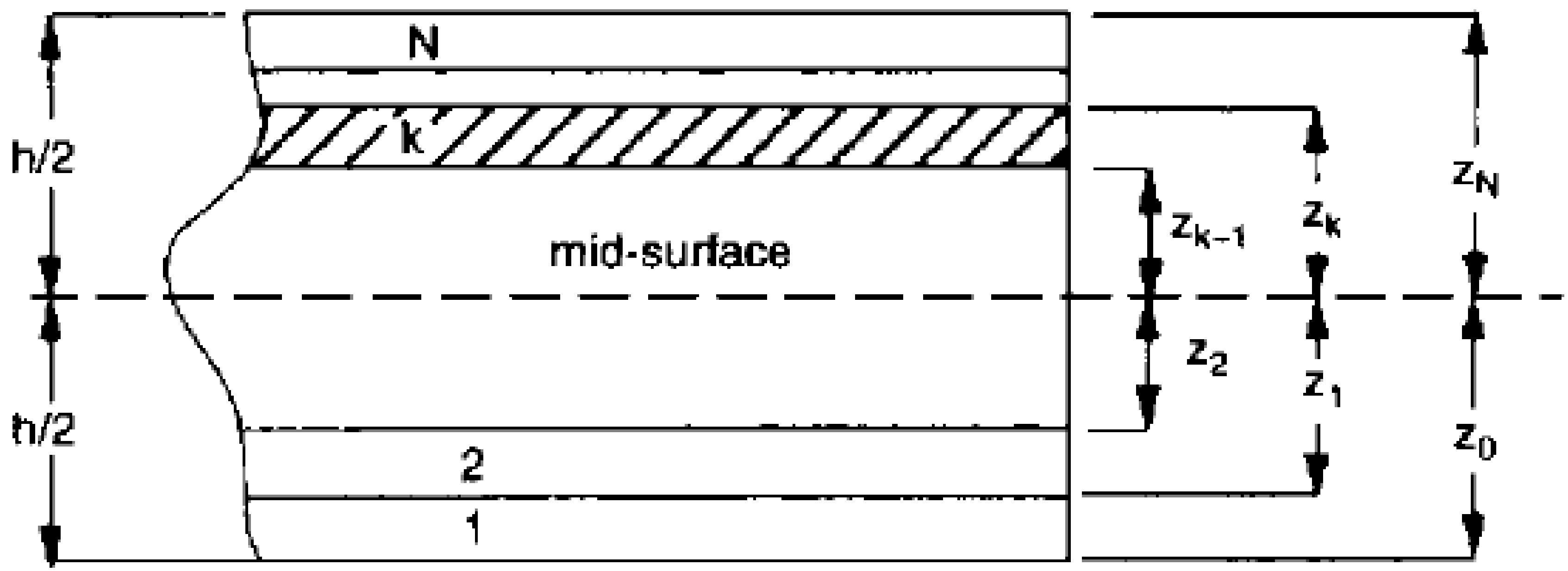


Figure: Laminate stacking sequence nomenclatures

Alternate Formulation of A, B, D

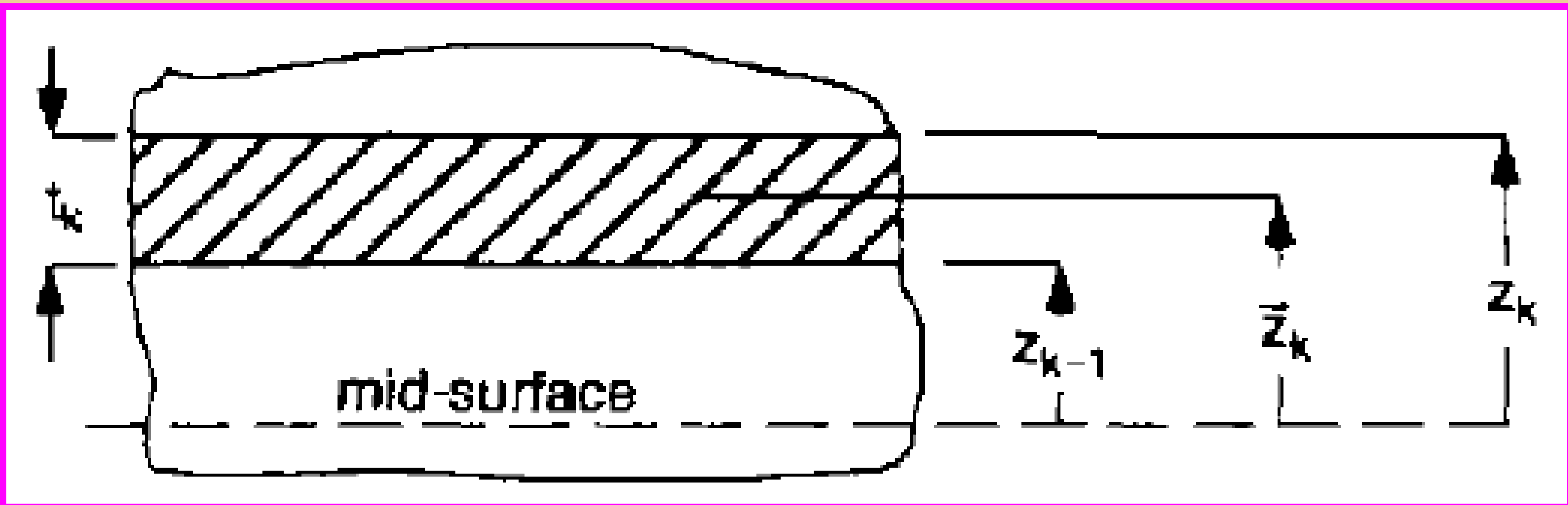


Figure : Relationship of $\bar{Z}_k$ and t_k to Z_k and Z_{k-1}

Alternate Formulation of A, B, D

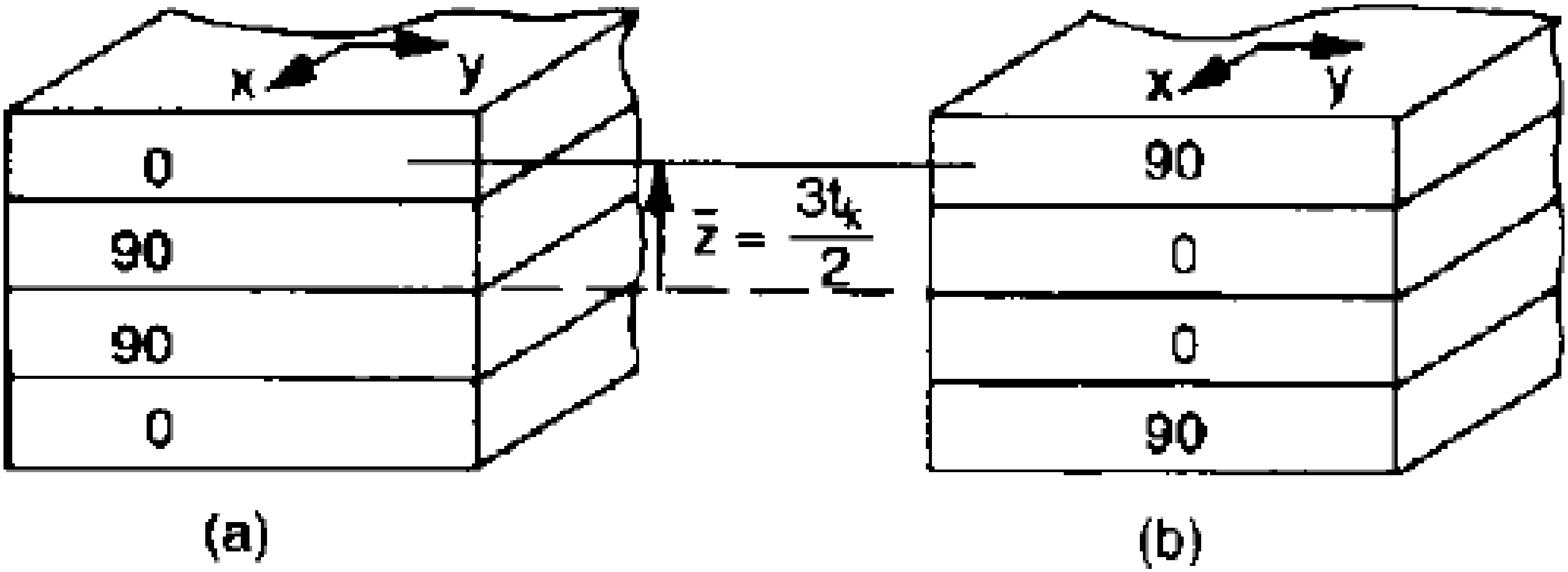


Figure M5.2.2.8 Two possible laminate stacking arrangements resulting in identical [A] matrices

Learning Unit M5.3

M5.3 Stress-Resultants in Laminate

Kirchhoff Hypothesis

1. Straight lines perpendicular to the midplane (i.e. transverse normals) before deformation remain straight after deformation.
2. The transverse normals do not experience elongation (i.e. they are inextensible).
3. The transverse normals rotate such that they remain perpendicular to the midplane after deformation.

Classical Lamination Theory

Follows all three Kirchhoff hypotheses.

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{u}^{0}(\mathbf{x},\mathbf{y},\mathbf{z}) - \mathbf{z}\frac{\partial \mathbf{w}^{0}}{\partial \mathbf{x}}$$

$$\mathbf{v}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{v}^{0}(\mathbf{x},\mathbf{y},\mathbf{z}) - \mathbf{z}\frac{\partial \mathbf{w}^{0}}{\partial \mathbf{y}}$$

$$\mathbf{w}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{w}^{0}(\mathbf{x},\mathbf{y})$$

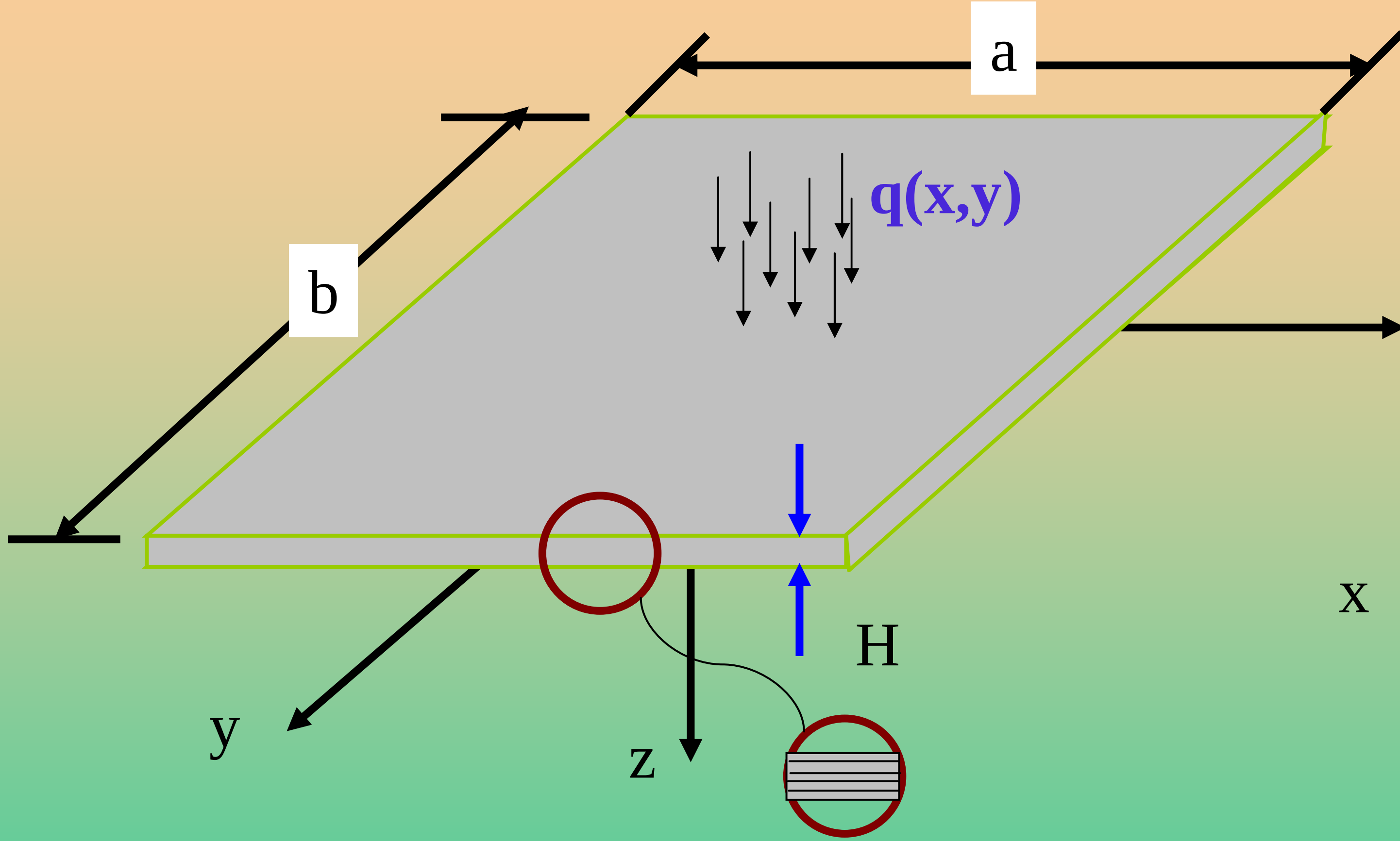
Second-Order Laminated Plate Theory

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{u}^0(\mathbf{x},\mathbf{y},\mathbf{z}) + \mathbf{z}\phi_x(\mathbf{x},\mathbf{y}) + \mathbf{z}^2\psi_x(\mathbf{x},\mathbf{y})$$

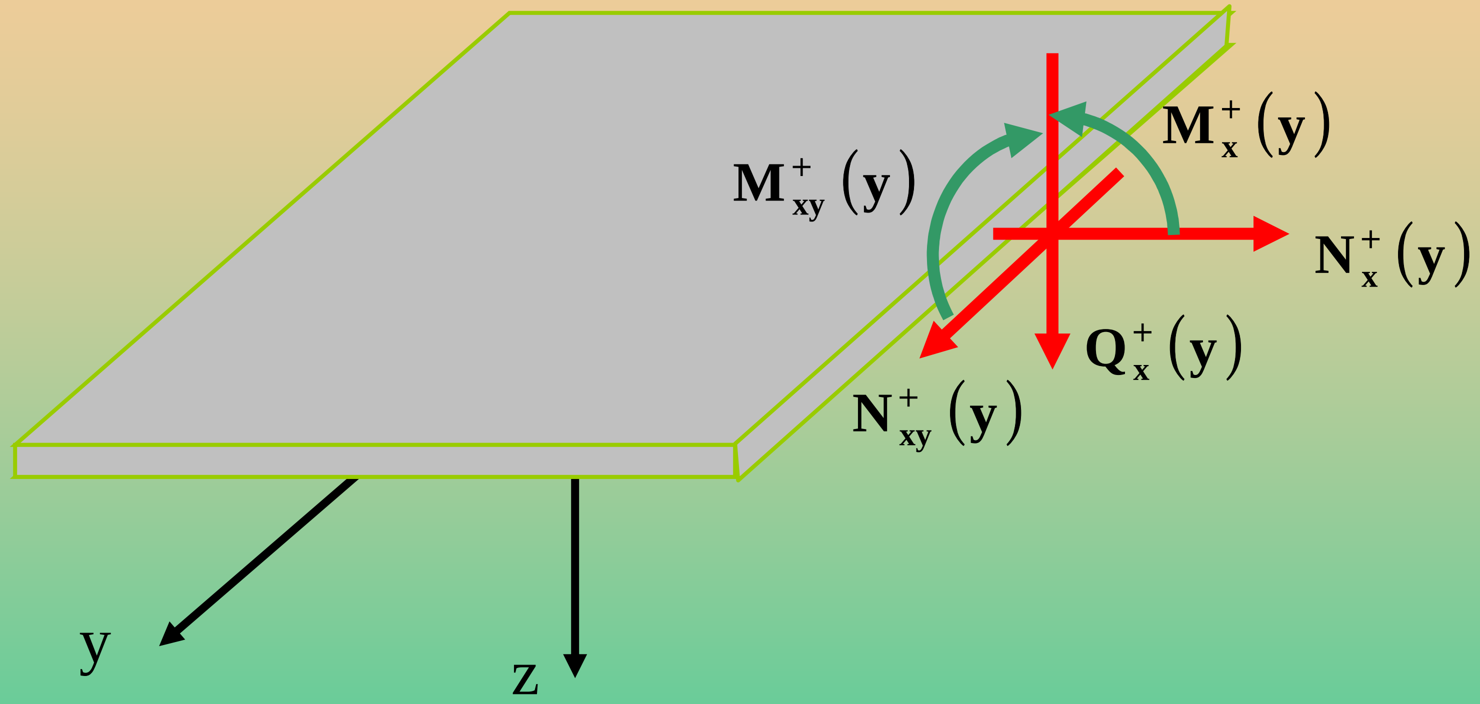
$$\mathbf{v}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{v}^0(\mathbf{x},\mathbf{y},\mathbf{z}) + \mathbf{z}\phi_x(\mathbf{x},\mathbf{y}) + \mathbf{z}^2\psi_x(\mathbf{x},\mathbf{y})$$

$$\mathbf{w}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{w}^0(\mathbf{x},\mathbf{y})$$

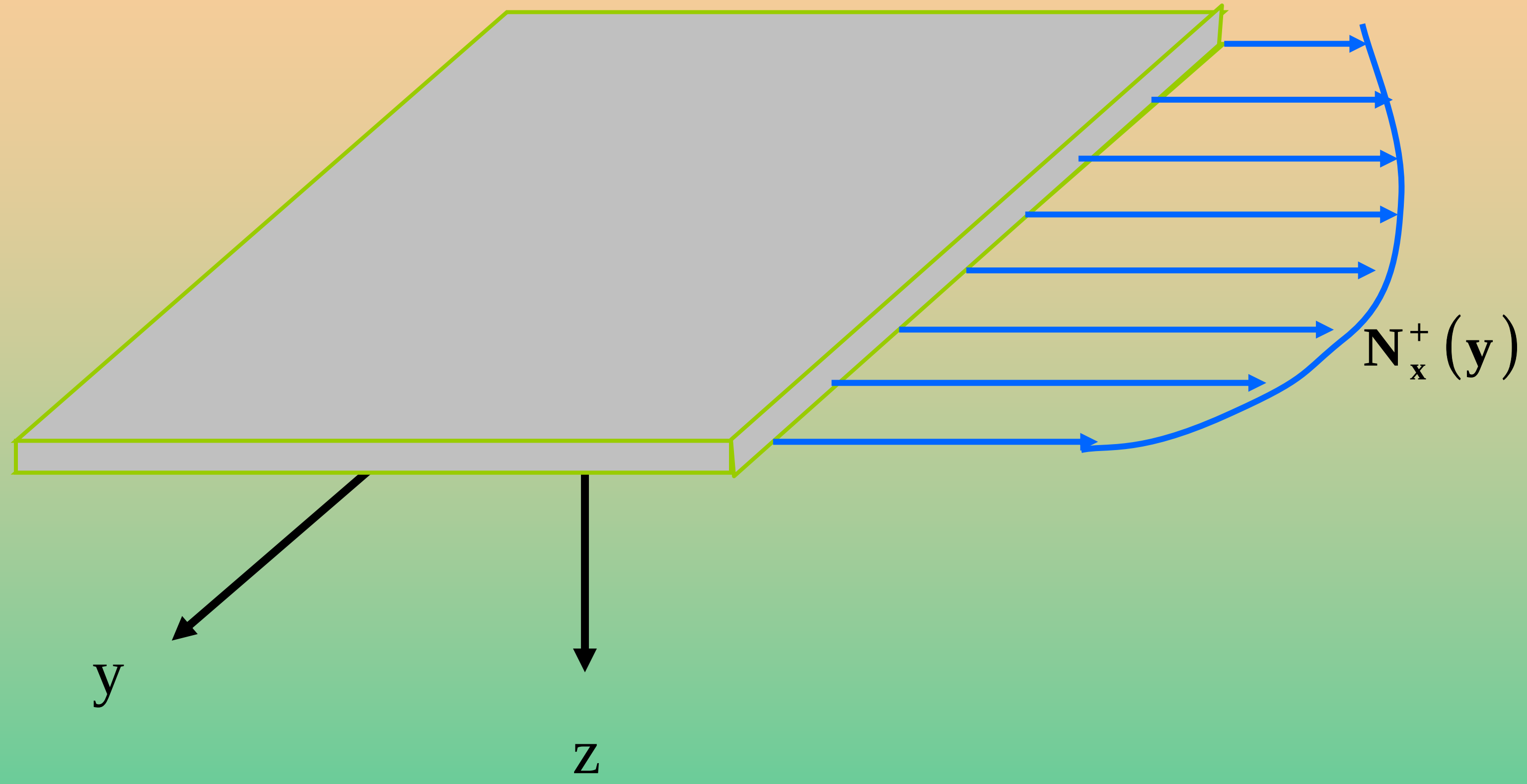
FSDT



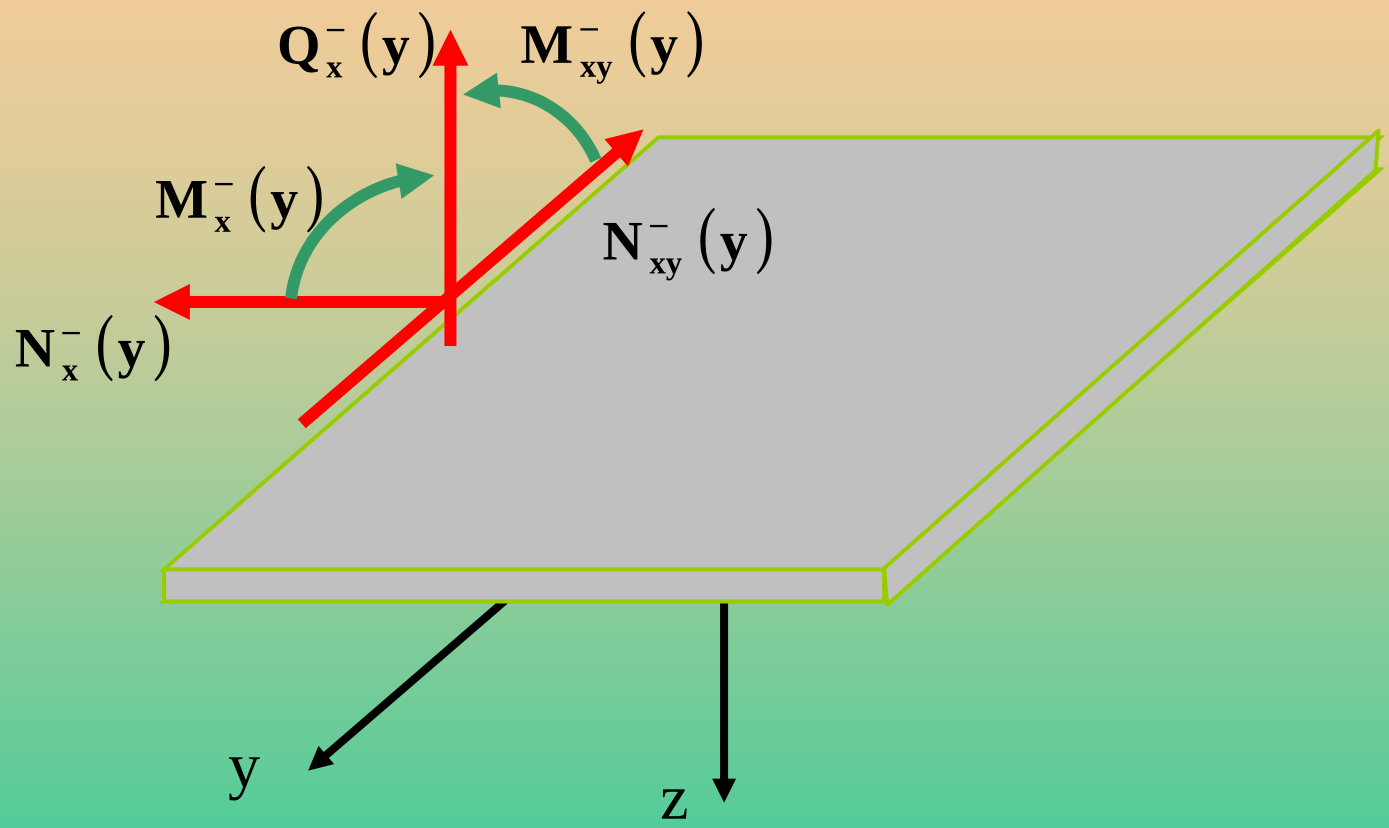
Stress Resultants at right edge



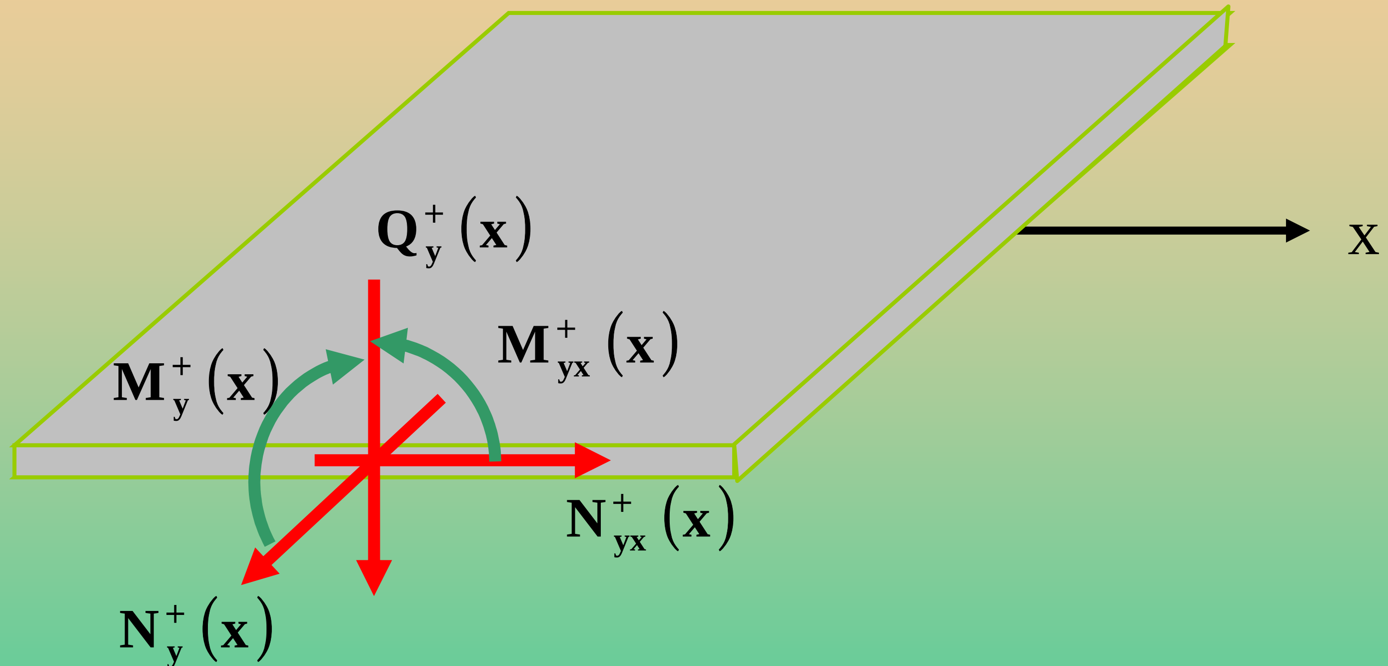
Variation of N_x



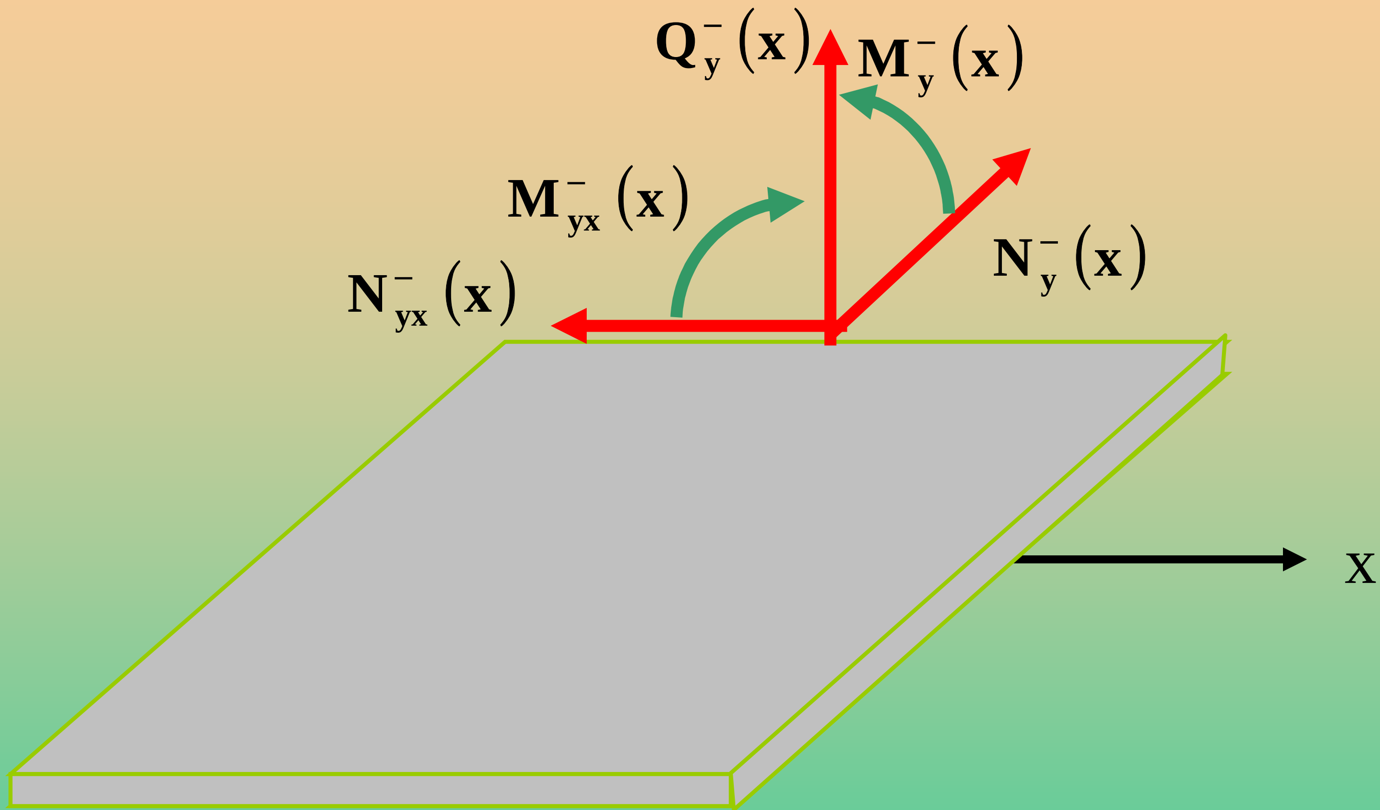
Stress Resultants at left edge



Stress Resultants at front edge



Stress Resultants at back edge



Stress Resultants

Normal force resultants :

$$N_x^+(y) = \int_{-H/2}^{H/2} \sigma_x\left(+\frac{a}{2}, y, z\right)dz$$

$$N_x^-(y) = \int_{-H/2}^{H/2} \sigma_x\left(-\frac{a}{2}, y, z\right)dz$$

$$N_y^+(x) = \int_{-H/2}^{H/2} \sigma_y\left(x, +\frac{b}{2}, z\right)dz$$

$$N_y^-(x) = \int_{-H/2}^{H/2} \sigma_y\left(x, -\frac{b}{2}, z\right)dz$$

Normal force resultants :

$$N_x^+(y) = \int_{-H/2}^{H/2} \sigma_x(a, y, z)dz$$

$$N_x^-(y) = \int_{-H/2}^{H/2} \sigma_x(0, y, z)dz$$

$$N_y^+(x) = \int_{-H/2}^{H/2} \sigma_y(x, b, z)dz$$

$$N_y^-(x) = \int_{-H/2}^{H/2} \sigma_y(x, 0, z)dz$$

Continued....

In - plane shear force resultants :

$$N_{xy}^{+}(y)=\int_{-H/2}^{H/2}\sigma_{xy}\left(+\frac{a}{2},y,z\right)dz$$

$$N_{xy}^{-}(y)=\int_{-H/2}^{H/2}\sigma_{xy}\left(-\frac{a}{2},y,z\right)dz$$

$$N_{yx}^{+}(x)=\int_{-H/2}^{H/2}\sigma_{yx}\left(x,+\frac{b}{2},z\right)dz$$

$$N_{yx}^{-}(x)=\int_{-H/2}^{H/2}\sigma_{yx}\left(x,-\frac{b}{2},z\right)dz$$

Continued….

Bending moment resultants :

$$\mathbf{M_x^+(y)=\int_{-H/2}^{H/2}\sigma_x\left(+\frac{a}{2},y,z\right)zdz}$$

$$\mathbf{M_x^-(y)=\int_{-H/2}^{H/2}\sigma_x\left(-\frac{a}{2},y,z\right)zdz}$$

$$\mathbf{M_y^+(x)=\int_{-H/2}^{H/2}\sigma_y\left(x,+\frac{b}{2},z\right)zdz}$$

$$\mathbf{M_y^-(x)=\int_{-H/2}^{H/2}\sigma_y\left(x,-\frac{b}{2},z\right)zdz}$$

Continued....

Twisting moment resultants :

$$M_{xy}^{+}(y) = \int_{-H/2}^{H/2} \sigma_{xy}\left(+\frac{a}{2}, y, z\right) z dz$$

$$M_{xy}^{-}(y) = \int_{-H/2}^{H/2} \sigma_{xy}\left(-\frac{a}{2}, y, z\right) z dz$$

$$M_{yx}^{+}(x) = \int_{-H/2}^{H/2} \sigma_{xy}\left(x, +\frac{b}{2}, z\right) z dz$$

$$M_{yx}^{-}(x) = \int_{-H/2}^{H/2} \sigma_{xy}\left(x, -\frac{b}{2}, z\right) z dz$$

Continued....

Transverse shear force resultants :

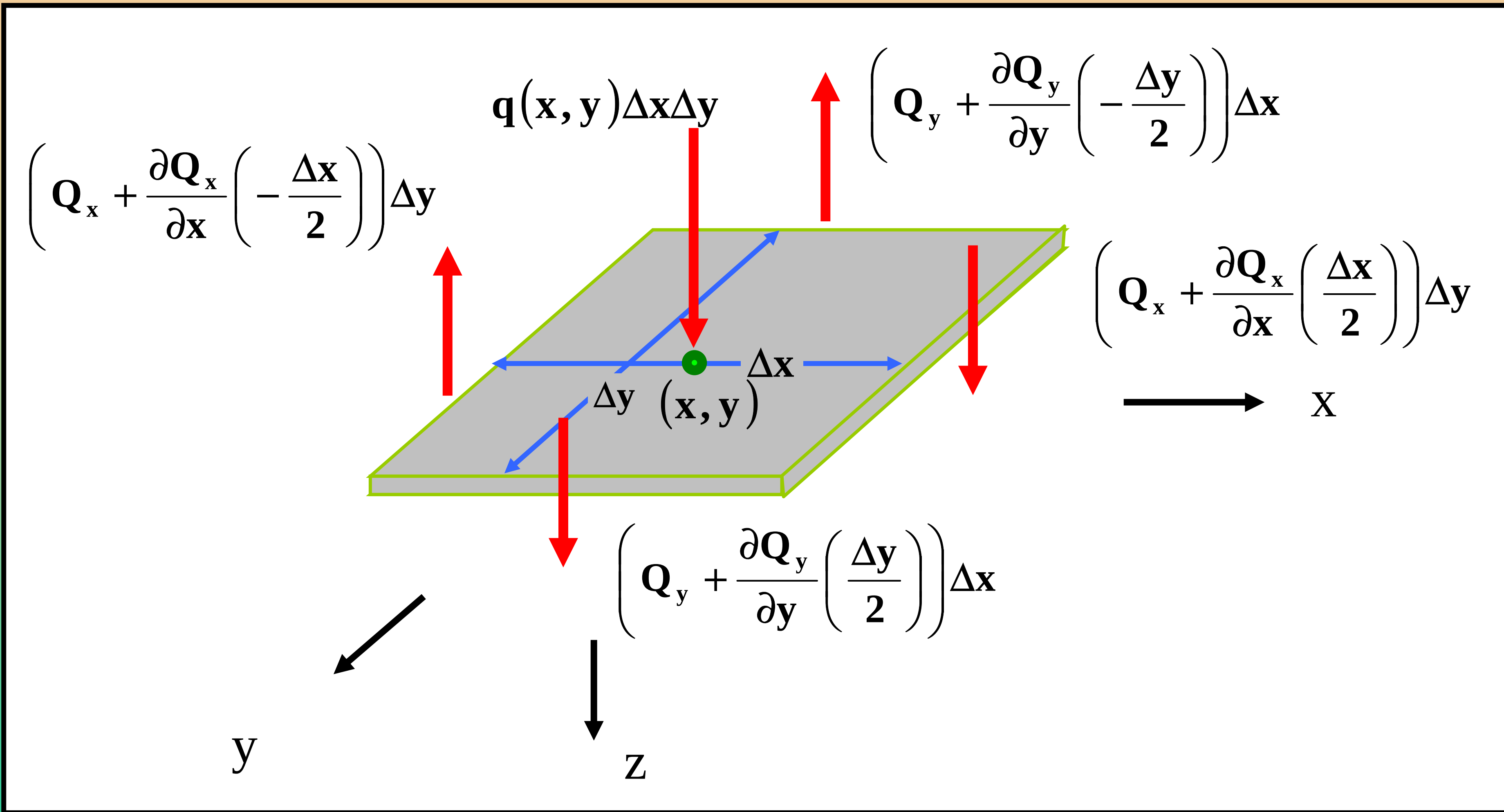
$$Q_x^+(y) = \int_{-H/2}^{H/2} \sigma_{xz}\left(+\frac{a}{2}, y, z\right) dz$$

$$Q_x^-(y) = \int_{-H/2}^{H/2} \sigma_{xz}\left(-\frac{a}{2}, y, z\right) dz$$

$$Q_y^+(x) = \int_{-H/2}^{H/2} \sigma_{yz}\left(x, +\frac{b}{2}, z\right) dz$$

$$Q_y^-(x) = \int_{-H/2}^{H/2} \sigma_{yz}\left(x, -\frac{b}{2}, z\right) dz$$

Equilibrium



Continued....

$$\sum F_z = 0$$

$$\left(Q_x + \frac{\partial Q_x}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y + \left(Q_y + \frac{\partial Q_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$$

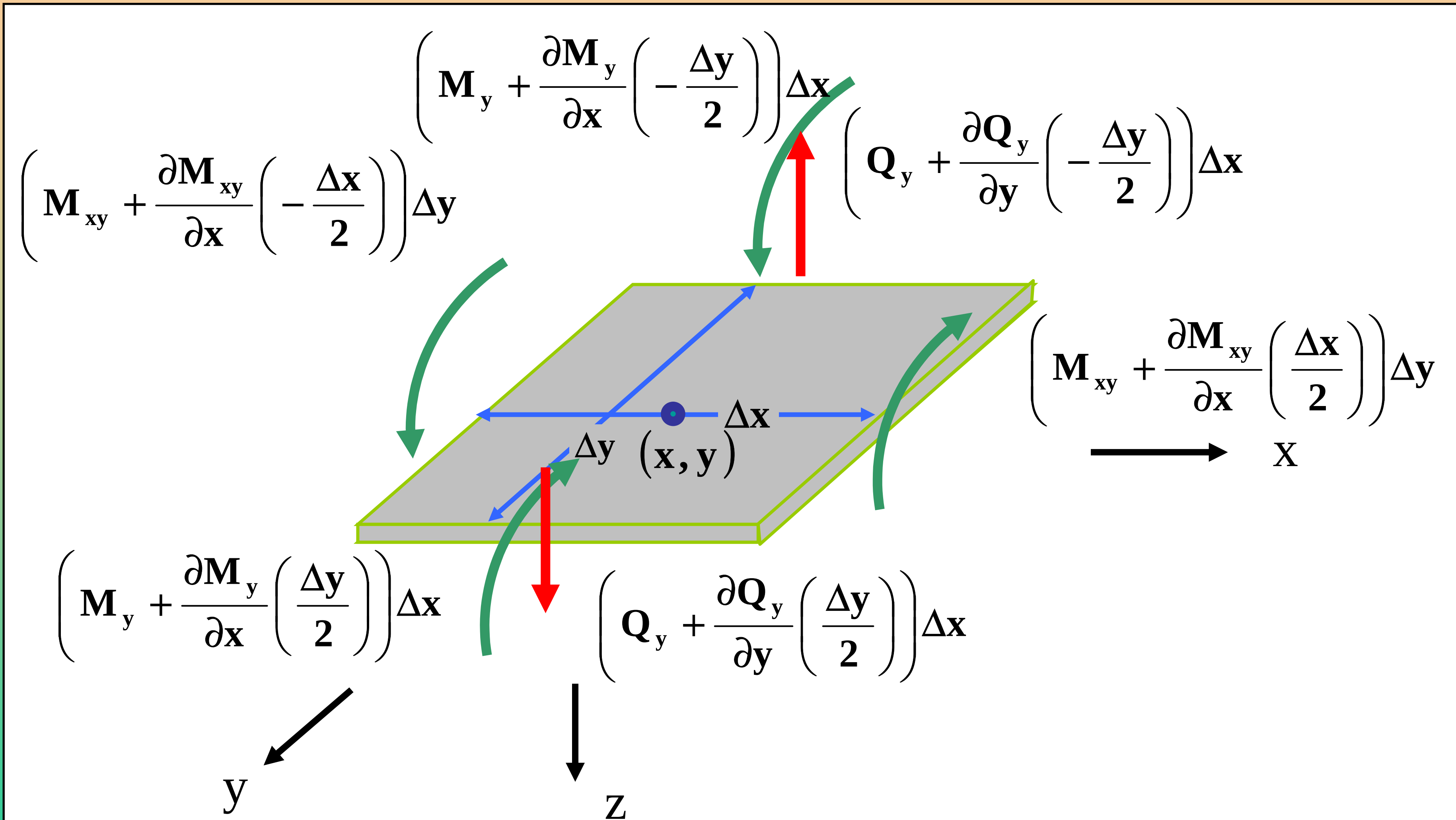
$$-\left(Q_x + \frac{\partial Q_x}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y$$

$$-\left(Q_y + \frac{\partial Q_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x + q\Delta x\Delta y = 0$$

Continued....

$$\frac{\partial \mathbf{Q_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{Q_y}}{\partial \mathbf{y}} + \mathbf{q} = \mathbf{0}$$

Equilibrium - Moments about x axis



Continued....

$$\sum M_x = 0$$

$$\left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y$$

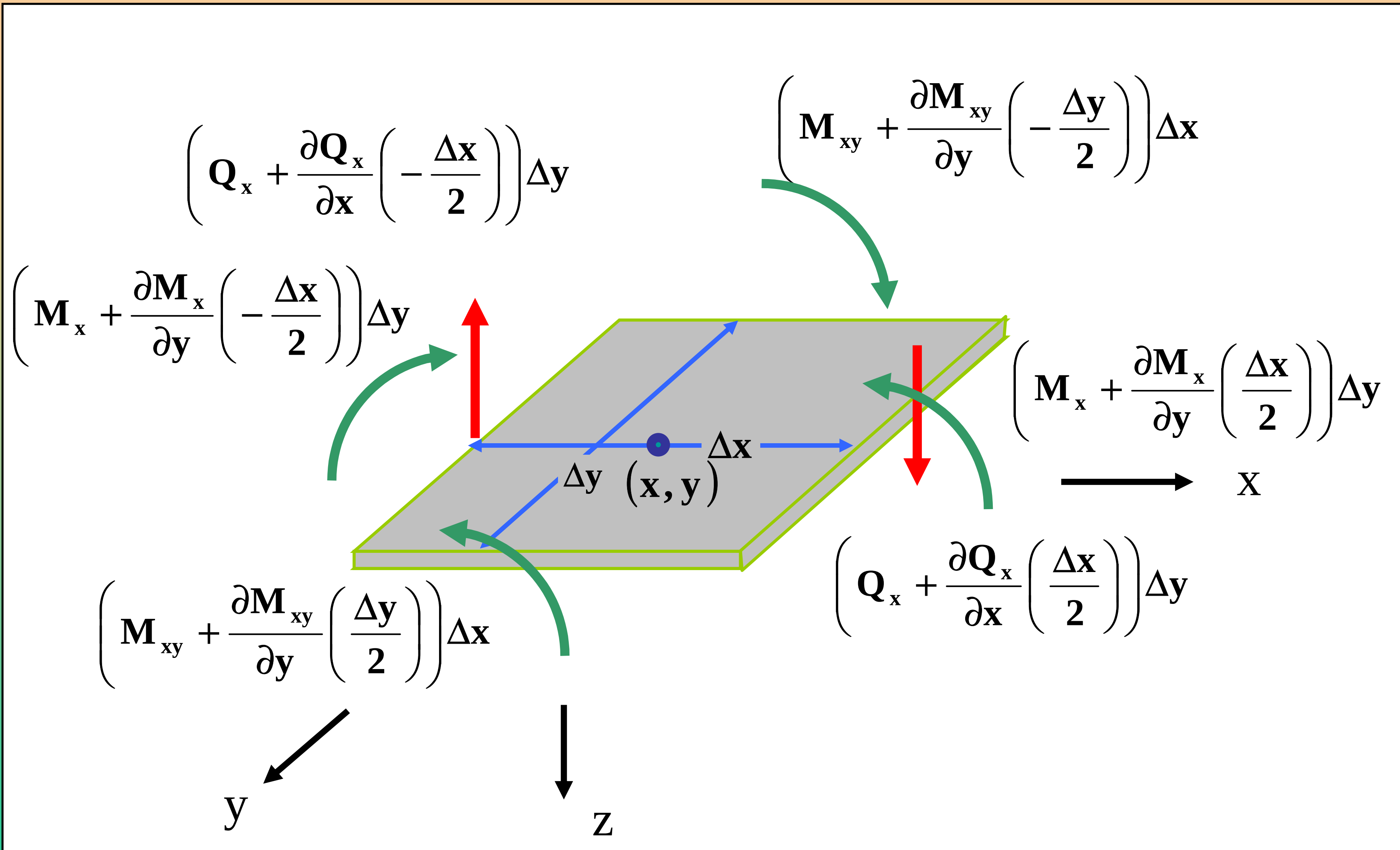
$$+\left(M_y + \frac{\partial M_y}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x - \left(M_y + \frac{\partial M_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$$

$$+\left(Q_y + \frac{\partial Q_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x\left(\frac{\Delta y}{2}\right) + \left(Q_y + \frac{\partial Q_y}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x\left(\frac{\Delta y}{2}\right) = 0$$

Continued....

$$\frac{\partial \mathbf{M}_{\mathbf{y}}}{\partial \mathbf{y}} + \frac{\partial \mathbf{M}_{\mathbf{xy}}}{\partial \mathbf{x}} = \mathbf{Q}_{\mathbf{y}}$$

Equilibrium - Moments about y-axis



Continued….

$$\sum M_y = 0$$

$$\left(M_{xy} + \frac{\partial M_{xy}}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x - \left(M_{xy} + \frac{\partial M_{xy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$$

$$+\left(M_x + \frac{\partial M_x}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y - \left(M_x + \frac{\partial M_x}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta x$$

$$+\left(Q_x + \frac{\partial Q_x}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y\left(\frac{\Delta y}{2}\right) + \left(Q_x + \frac{\partial Q_x}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y\left(\frac{\Delta x}{2}\right) = 0$$

Continued….

$$\frac{\partial \mathbf{M_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{M_{xy}}}{\partial \mathbf{y}} = \mathbf{Q_x}$$

Continued....

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

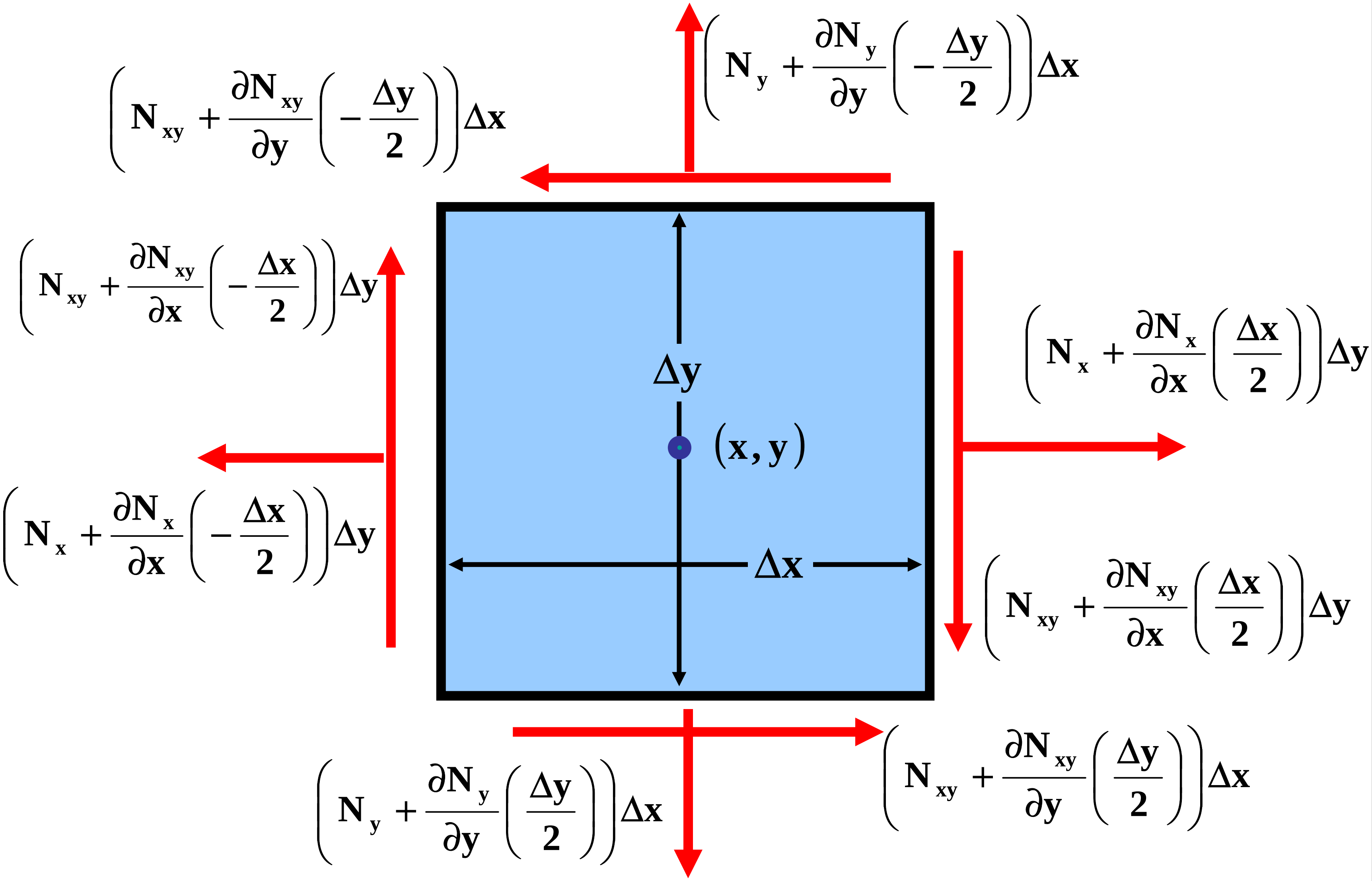
$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

Continued....

$$\frac{\partial\left(\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right)}{\partial x}+\frac{\partial\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)}{\partial y}+q=0$$

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0$$


$\left(N_y + \frac{\partial N_y}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x$
$\left(N_{xy} + \frac{\partial N_{xy}}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x$
$\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y$
$\left(N_x + \frac{\partial N_x}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y$
Δy
(x, y)
$\left(N_x + \frac{\partial N_x}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y$
Δx
$\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y$
$\left(N_{xy} + \frac{\partial N_{xy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$
$\left(N_y + \frac{\partial N_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$

Equilibrium

$$\sum F_x = 0$$

$$\left(N_x + \frac{\partial N_x}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x$$

$$-\left(N_x + \frac{\partial N_x}{\partial x}\left(-\frac{\Delta x}{2}\right)\right)\Delta y$$

$$-\left(N_{xy} + \frac{\partial N_{xy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x = 0$$

Equilibrium

$$\frac{\partial \mathbf{N}_{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{\mathbf{xy}}}{\partial \mathbf{y}} = \mathbf{0}$$

$$\sum F_y = 0$$

$$\left(N_y + \frac{\partial N_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right)\Delta x + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y$$

$$-\left(N_y + \frac{\partial N_y}{\partial y}\left(-\frac{\Delta y}{2}\right)\right)\Delta x$$

$$-\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}\left(\frac{\Delta x}{2}\right)\right)\Delta y = 0$$

$$\frac{\partial \mathbf{N}_{\mathbf{xy}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{\mathbf{y}}}{\partial \mathbf{y}} = \mathbf{0}$$

$$\sum \mathbf{M}_z = 0$$

$$-\left(\mathbf{N}_{xy}+\frac{\partial \mathbf{N}_{xy}}{\partial \mathbf{y}}\left(-\frac{\Delta \mathbf{y}}{2}\right)\right)\Delta \mathbf{x}\left(\frac{\Delta \mathbf{y}}{2}\right)-\left(\mathbf{N}_{xy}+\frac{\partial \mathbf{N}_{xy}}{\partial \mathbf{y}}\left(\frac{\Delta \mathbf{y}}{2}\right)\right)\Delta \mathbf{x}\left(\frac{\Delta \mathbf{y}}{2}\right)$$

$$-\left(\mathbf{N}_{xy}+\frac{\partial \mathbf{N}_{xy}}{\partial \mathbf{x}}\left(-\frac{\Delta \mathbf{x}}{2}\right)\right)\Delta \mathbf{y}\left(\frac{\Delta \mathbf{x}}{2}\right)-\left(\mathbf{N}_{xy}+\frac{\partial \mathbf{N}_{xy}}{\partial \mathbf{x}}\left(-\frac{\Delta \mathbf{x}}{2}\right)\right)\Delta \mathbf{y}\left(\frac{\Delta \mathbf{y}}{2}\right)\equiv 0$$

Plate Equilibrium Equations

$$\frac{\partial \mathbf{N_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N_{xy}}}{\partial \mathbf{y}} = 0$$

$$\frac{\partial \mathbf{N_{xy}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N_y}}{\partial \mathbf{y}} = 0$$

$$\frac{\partial^2 \mathbf{M_x}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{M_{xy}}}{\partial \mathbf{x} \partial \mathbf{y}} + \frac{\partial^2 \mathbf{M_y}}{\partial \mathbf{y}^2} + \mathbf{q} = 0$$

Displacement Based Equations

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0 + B_{11}\kappa_x + B_{12}\kappa_y + B_{16}\kappa_{xy}$$

$$N_x = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y} + A_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - B_{11}\frac{\partial^2 w^0}{\partial x^2} - B_{12}\frac{\partial^2 w^0}{\partial y^2} - 2B_{16}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equations

$$N_y = A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\gamma_{xy}^0 + B_{12}\kappa_x + B_{22}\kappa_y + B_{26}\kappa_{xy}$$

$$N_y = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y} + A_{26}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - B_{12}\frac{\partial^2 w^0}{\partial x^2} - B_{22}\frac{\partial^2 w^0}{\partial y^2} - 2B_{26}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equations

$$N_{xy} = A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0 + A_{66}\gamma_{xy}^0 + B_{16}\kappa_x + B_{26}\kappa_y + B_{66}\kappa_{xy}$$

$$N_{xy} = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - B_{16}\frac{\partial^2 w^0}{\partial x^2} - B_{26}\frac{\partial^2 w^0}{\partial y^2} - 2B_{66}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equations

$$M_x = B_{11}\varepsilon_x^0 + B_{12}\varepsilon_y^0 + B_{16}\gamma_{xy}^0 + D_{11}\kappa_x + D_{12}\kappa_y + D_{16}\kappa_{xy}$$

$$M_x = B_{11}\frac{\partial u^0}{\partial x} + B_{12}\frac{\partial v^0}{\partial y} + B_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equations

$$M_y = B_{12}\varepsilon_x^0 + B_{22}\varepsilon_y^0 + B_{26}\gamma_{xy}^0 + D_{12}\kappa_x + D_{22}\kappa_y + D_{26}\kappa_{xy}$$

$$M_y = B_{12}\frac{\partial u^0}{\partial x} + B_{22}\frac{\partial v^0}{\partial y} + B_{26}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - D_{12}\frac{\partial^2 w^0}{\partial x^2} - D_{22}\frac{\partial^2 w^0}{\partial y^2} - 2D_{26}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equations

$$M_{xy} = B_{16}\varepsilon_x^0 + B_{26}\varepsilon_y^0 + B_{66}\gamma_{xy}^0 + D_{16}\kappa_x + D_{26}\kappa_y + D_{66}\kappa_{xy}$$

$$M_{xy} = B_{16}\frac{\partial u^0}{\partial x} + B_{26}\frac{\partial v^0}{\partial y} + B_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - D_{16}\frac{\partial^2 w^0}{\partial x^2} - D_{26}\frac{\partial^2 w^0}{\partial y^2} - 2D_{66}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Plate Equilibrium Equations

$$\frac{\partial \mathbf{N}_{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{\mathbf{xy}}}{\partial \mathbf{y}} = \mathbf{0}$$

$$\frac{\partial \mathbf{N}_{\mathbf{xy}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{\mathbf{y}}}{\partial \mathbf{y}} = \mathbf{0}$$

$$\frac{\partial^2 \mathbf{M}_{\mathbf{x}}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{M}_{\mathbf{xy}}}{\partial \mathbf{x} \partial \mathbf{y}} + \frac{\partial^2 \mathbf{M}_{\mathbf{y}}}{\partial \mathbf{y}^2} + \mathbf{q} = \mathbf{0}$$

Displacement Based Equilibrium Equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$N_x = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y} + A_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

$$- B_{11}\frac{\partial^2 w^0}{\partial x^2} - B_{12}\frac{\partial^2 w^0}{\partial y^2} - 2B_{16}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

$$N_{xy} = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

$$- B_{16}\frac{\partial^2 w^0}{\partial x^2} - B_{26}\frac{\partial^2 w^0}{\partial y^2} - 2B_{66}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Displacement Based Equilibrium Equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2} + A_{12}\frac{\partial^2 v^0}{\partial y \partial x} + A_{16}\left(\frac{\partial^2 u^0}{\partial y \partial x} + \frac{\partial^2 v^0}{\partial x^2}\right)$$

$$- B_{11}\frac{\partial^3 w^0}{\partial x^3} - B_{12}\frac{\partial^3 w^0}{\partial x \partial y^2} - 2B_{16}\left(\frac{\partial^3 w^0}{\partial x^2 \partial y}\right)$$

$$+ A_{16}\frac{\partial^2 u^0}{\partial x \partial y} + A_{26}\frac{\partial^2 v^0}{\partial y^2} + A_{66}\left(\frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)$$

$$- B_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y} - B_{26}\frac{\partial^3 w^0}{\partial y^3} - 2B_{66}\left(\frac{\partial^3 w^0}{\partial x \partial y^2}\right)$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2} + 2A_{16}\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 u^0}{\partial y^2}$$
$$+ A_{16}\frac{\partial^2 v^0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v^0}{\partial x \partial y} + + A_{26}\frac{\partial^2 v^0}{\partial y^2}$$
$$- B_{11}\frac{\partial^3 w^0}{\partial x^3} - 3B_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y} - (B_{12} + 2B_{66})\frac{\partial^3 w^0}{\partial x \partial y^2} - B_{26}\frac{\partial^3 w^0}{\partial y^3} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u^0}{\partial x \partial y} + A_{26}\frac{\partial^2 u^0}{\partial y^2}$$

$$+ A_{66}\frac{\partial^2 v^0}{\partial x^2} + 2A_{26}\frac{\partial^2 v^0}{\partial x \partial y} + A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$- B_{16}\frac{\partial^3 w^0}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 w^0}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - B_{22}\frac{\partial^3 w^0}{\partial y^3} = 0$$

$$\frac{\partial^2 \mathbf{M}_x}{\partial \mathbf{x}^2} + 2\frac{\partial^2 \mathbf{M}_{xy}}{\partial \mathbf{x}\partial \mathbf{y}} + \frac{\partial^2 \mathbf{M}_y}{\partial \mathbf{y}^2} + \mathbf{q} = 0$$

$$\mathbf{D}_{11}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^4} + 4\mathbf{D}_{16}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^3\partial \mathbf{y}} + 2\left(\mathbf{D}_{12} + 2\mathbf{D}_{66}\right)\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^2\partial \mathbf{y}^2}$$

$$+ 4\mathbf{D}_{26}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}\partial \mathbf{y}^3} + \mathbf{D}_{22}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{y}^4}$$

$$- \mathbf{B}_{11}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}^3} - 3\mathbf{B}_{16}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}^2\partial \mathbf{y}} - \left(\mathbf{B}_{12} + 2\mathbf{B}_{66}\right)\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}\partial \mathbf{y}^2} - \mathbf{B}_{26}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{y}^3}$$

$$- \mathbf{B}_{16}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}^3} - \left(\mathbf{B}_{12} + 2\mathbf{B}_{66}\right)\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}^2\partial \mathbf{y}} - 3\mathbf{B}_{26}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}\partial \mathbf{y}^2} - \mathbf{B}_{22}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{y}^3} = \mathbf{q}$$

Symmetric Laminates

$$B_{ij} = 0$$

$$\mathbf{A}_{11}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}^2}+2\mathbf{A}_{16}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}\partial\mathbf{x}}+\mathbf{A}_{66}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}^2}+\mathbf{A}_{16}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}^2}$$

$$+\left(\mathbf{A}_{12}+\mathbf{A}_{66}\right)\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}\partial\mathbf{x}}+\mathbf{A}_{26}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}^2}-\mathbf{B}_{11}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}^3}$$

$$-3\mathbf{B}_{16}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}^2\partial\mathbf{y}}-\left(\mathbf{B}_{12}+2\mathbf{B}_{66}\right)\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}\partial\mathbf{y}^2}-\mathbf{B}_{26}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{y}^3}=0$$

$$\mathbf{A}_{11}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}^2}+2\mathbf{A}_{16}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}\partial\mathbf{y}}+\mathbf{A}_{66}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}^2}+\mathbf{A}_{16}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}^2}$$

$$+\left(\mathbf{A}_{12}+\mathbf{A}_{66}\right)\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}\partial\mathbf{y}}+\mathbf{A}_{26}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}^2}=0$$

$$\mathbf{A}_{16}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}^2}+\left(\mathbf{A}_{12}+\mathbf{A}_{66}\right)\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}\partial\mathbf{x}}+\mathbf{A}_{26}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}^2}+\mathbf{A}_{66}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}^2}$$

$$+2\mathbf{A}_{26}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}\partial\mathbf{x}}+\mathbf{A}_{22}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}^2}-\mathbf{B}_{16}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}^3}-\left(\mathbf{B}_{12}+2\mathbf{B}_{66}\right)\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}^2\partial\mathbf{y}}$$

$$-3\mathbf{B}_{26}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{x}\partial\mathbf{y}^2}-\mathbf{B}_{22}\frac{\partial^3\mathbf{w}^0}{\partial\mathbf{y}^3}=0$$

$$\mathbf{A}_{16}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}^2}+\left(\mathbf{A}_{12}+\mathbf{A}_{66}\right)\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{x}\partial\mathbf{y}}+\mathbf{A}_{26}\frac{\partial^2\mathbf{u}^0}{\partial\mathbf{y}^2}$$

$$+\mathbf{A}_{66}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}^2}+2\mathbf{A}_{26}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{x}\partial\mathbf{y}}+\mathbf{A}_{22}\frac{\partial^2\mathbf{v}^0}{\partial\mathbf{y}^2}=0$$

$$\mathbf{D}_{11}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^4}+4\mathbf{D}_{16}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^3\partial \mathbf{y}}+2(\mathbf{D}_{12}+2\mathbf{D}_{66})\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^2\partial \mathbf{y}^2}+4\mathbf{D}_{26}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}\partial \mathbf{y}^3}$$

$$+\mathbf{D}_{22}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{y}^4}-\mathbf{B}_{11}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}^3}-3\mathbf{B}_{16}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}^2\partial \mathbf{y}}-(\mathbf{B}_{12}+2\mathbf{B}_{66})\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{x}\partial \mathbf{y}^2}-\mathbf{B}_{26}\frac{\partial^3 \mathbf{u}^0}{\partial \mathbf{y}^3}$$

$$-\mathbf{B}_{16}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}^3}-(\mathbf{B}_{12}+2\mathbf{B}_{66})\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}^2\partial \mathbf{y}}-3\mathbf{B}_{26}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{x}\partial \mathbf{y}^2}-\mathbf{B}_{22}\frac{\partial^3 \mathbf{v}^0}{\partial \mathbf{y}^3}=\mathbf{q}$$

$$\mathbf{D}_{11}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^4}+4\mathbf{D}_{16}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^3\partial \mathbf{y}}+2(\mathbf{D}_{12}+2\mathbf{D}_{66})\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^2\partial \mathbf{y}^2}$$

$$+4\mathbf{D}_{26}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}\partial \mathbf{y}^3}+\mathbf{D}_{22}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{y}^4}=0$$

Right Edge

i. *Either* $N_x^+ = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y} + A_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or u^0 *must be specified.*

ii. *Either* $N_{xy}^+ = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_x^+ + \frac{\partial M_{xy}^+}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - 2D_{26}\frac{\partial^3 w^0}{\partial y^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2} - 4D_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_x^+ = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Left Edge

i. *Either* $N_x^- = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y} + A_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or u^0 *must be specified.*

ii. *Either* $N_{xy}^- = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_x^- + \frac{\partial M_{xy}^-}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - 2D_{26}\frac{\partial^3 w^0}{\partial y^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2} - 4D_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_x^- = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Front Edge

i. *Either* $$N_y^+ = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y} + A_{26}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

or v^0 *must be specified.*

ii. *Either* $$N_{yx}^+ = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

or u^0 *must be specified.*

iii. *Either* $$Q_y^+ + \frac{\partial M_{yx}^+}{\partial y} = -2D_{16}\frac{\partial^3 w^0}{\partial x^3} - D_{22}\frac{\partial^3 w^0}{\partial y^3} - 4D_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w^0}{\partial x \partial y^2}$$

or w^0 *must be specified.*

iv. *Either* $$M_y^+ = -D_{12}\frac{\partial^2 w^0}{\partial x^2} - D_{22}\frac{\partial^2 w^0}{\partial y^2} - 2D_{26}\frac{\partial^2 w^0}{\partial x \partial y}$$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Back Edge

i. *Either* $N_y^- = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y} + A_{26}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

ii. *Either* $N_{yx}^- = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or u^0 *must be specified.*

iii. *Either* $Q_y^- + \frac{\partial M_{yx}^-}{\partial y} = -2D_{16}\frac{\partial^3 w^0}{\partial x^3} - D_{22}\frac{\partial^3 w^0}{\partial y^3} - 4D_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2}$

or w^0 *must be specified.*

iv. *Either* $M_y^- = -D_{12}\frac{\partial^2 w^0}{\partial x^2} - D_{22}\frac{\partial^2 w^0}{\partial y^2} - 2D_{26}\frac{\partial^2 w^0}{\partial x \partial y}$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Symmetric Balanced Laminates

$$\mathbf{B_{ij} = 0}$$

$$\mathbf{A_{16} = 0}$$

$$\mathbf{A_{26} = 0}$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

$$\mathbf{A}_{11}\frac{\partial^2 \mathbf{u}^0}{\partial \mathbf{x}^2}+\mathbf{A}_{66}\frac{\partial^2 \mathbf{u}^0}{\partial \mathbf{y}^2}+(\mathbf{A}_{12}+\mathbf{A}_{66})\frac{\partial^2 \mathbf{v}^0}{\partial \mathbf{x}\partial \mathbf{y}}=0$$

$$(\mathbf{A}_{12}+\mathbf{A}_{66})\frac{\partial^2 \mathbf{u}^0}{\partial \mathbf{x}\partial \mathbf{y}}+\mathbf{A}_{66}\frac{\partial^2 \mathbf{v}^0}{\partial \mathbf{x}^2}+\mathbf{A}_{22}\frac{\partial^2 \mathbf{v}^0}{\partial \mathbf{y}^2}=0$$

$$\mathbf{D}_{11}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^4}+4\mathbf{D}_{16}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^3\partial \mathbf{y}}+2(\mathbf{D}_{12}+2\mathbf{D}_{66})\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}^2\partial \mathbf{y}^2}$$

$$+4\mathbf{D}_{26}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{x}\partial \mathbf{y}^3}+\mathbf{D}_{22}\frac{\partial^4 \mathbf{w}^0}{\partial \mathbf{y}^4}=\mathbf{q}$$

Right Edge

i. *Either* $N_x^+ = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y}$

or u^0 *must be specified.*

ii. *Either* $N_{xy}^+ = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_x^+ + \frac{\partial M_{xy}^+}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - 2D_{26}\frac{\partial^3 w^0}{\partial y^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2} - 4D_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_x^+ = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Left Edge

i. *Either* $N_x^- = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y}$

or u^0 *must be specified.*

ii. *Either* $N_{xy}^- = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_x^- + \frac{\partial M_{xy}^-}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - 2D_{26}\frac{\partial^3 w^0}{\partial y^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2} - 4D_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_x^- = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Front Edge

i. *Either* $$N_y^+ = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y}$$

or u^0 *must be specified.*

ii. *Either* $$N_{yx}^+ = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

or v^0 *must be specified.*

iii. *Either* $$Q_y^+ + \frac{\partial M_{yx}^+}{\partial y} = -2D_{16}\frac{\partial^3 w^0}{\partial x^3} - D_{22}\frac{\partial^3 w^0}{\partial y^3} - 4D_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w^0}{\partial x^2 \partial y}$$

or w^0 *must be specified.*

iv. *Either* $$M_y^+ = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Back Edge

i. *Either* $$N_y^- = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y}$$

or u^0 *must be specified.*

ii. *Either* $$N_{yx}^- = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

or v^0 *must be specified.*

iii. *Either* $$Q_y^- + \frac{\partial M_{yx}^-}{\partial y} = 2D_{16}\frac{\partial^3 w^0}{\partial x^3} - D_{22}\frac{\partial^3 w^0}{\partial y^3} - 4D_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w^0}{\partial x^2 \partial y}$$

or w^0 *must be specified.*

iv. *Either* $$M_y^- = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2} - 2D_{16}\frac{\partial^2 w^0}{\partial x \partial y}$$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Symmetric Cross-Ply Laminates

$$B_{ij} = 0$$

$$A_{16} = 0 \qquad D_{16} = 0$$

$$A_{26} = 0 \qquad D_{26} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

$$\mathbf{A_{11}\frac{\partial^2 u^0}{\partial x^2} + A_{66}\frac{\partial^2 u^0}{\partial y^2} + \left(A_{12} + A_{66}\right)\frac{\partial^2 v^0}{\partial x \partial y} = 0}$$

$$\mathbf{\left(A_{12} + A_{66}\right)\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 v^0}{\partial x^2} + A_{22}\frac{\partial^2 v^0}{\partial y^2} = 0}$$

$$\mathbf{D_{11}\frac{\partial^4 w^0}{\partial x^4 \partial y} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w^0}{\partial y^4} = q}$$

Right Edge

i. *Either* $$N_x^+ = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y}$$

or u^0 *must be specified.*

ii. *Either* $$N_{xy}^+ = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$$

or v^0 *must be specified.*

iii. *Either* $$Q_x^+ + \frac{\partial M_{xy}^+}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2}$$

or w^0 *must be specified.*

iv. *Either* $$M_x^+ = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2}$$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Left Edge

i. *Either* $N_x^- = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y}$

or u^0 *must be specified.*

ii. *Either* $N_{xy}^- = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_x^- + \frac{\partial M_{xy}^-}{\partial y} = -D_{11}\frac{\partial^3 w^0}{\partial x^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x \partial y^2}$

or w^0 *must be specified.*

iv. *Either* $M_x^- = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2}$

or $\frac{\partial w^0}{\partial x}$ *must be specified.*

Front Edge

i. *Either* $N_y^+ = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y}$

or u^0 *must be specified.*

ii. *Either* $N_{yx}^+ = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_y^+ + \frac{\partial M_{yx}^+}{\partial y} = -D_{22}\frac{\partial^3 w^0}{\partial y^3} - \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_y^+ = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2}$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Back Edge

i. *Either* $N_y^- = A_{12}\frac{\partial u^0}{\partial x} + A_{22}\frac{\partial v^0}{\partial y}$

or u^0 *must be specified.*

ii. *Either* $N_{yx}^- = A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)$

or v^0 *must be specified.*

iii. *Either* $Q_y^- + \frac{\partial M_{yx}^-}{\partial y} = -D_{22}\frac{\partial^3 w^0}{\partial y^3} - (D_{12} + 4D_{66})\frac{\partial^3 w^0}{\partial x^2 \partial y}$

or w^0 *must be specified.*

iv. *Either* $M_y^- = -D_{11}\frac{\partial^2 w^0}{\partial x^2} - D_{12}\frac{\partial^2 w^0}{\partial y^2}$

or $\frac{\partial w^0}{\partial y}$ *must be specified.*

Isotropic Plates

$$B_{ij} = 0$$

$$A_{11} = A_{22} = \frac{EH}{1-\nu^2} = A \qquad A_{12} = A_{21} = \nu\frac{EH}{1-\nu^2} = \nu A$$

$$A_{66} = \frac{EH}{2(1+\nu)} = \frac{1-\nu}{2}A \qquad A_{16} = A_{26} = 0$$

$$D_{11} = D_{22} = \frac{EH^3}{12(1-\nu^2)} = D \qquad D_{12} = D_{21} = \frac{\nu EH^3}{12(1-\nu^2)} = \nu D$$

$$D_{66} = \frac{EH^3}{24(1+\nu)} = \frac{1-\nu}{2}D \qquad D_{16} = D_{26} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

$$A\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)A\frac{\partial^2 u^0}{\partial y^2}+\left(\nu A+\frac{1-\nu}{2}A\right)\frac{\partial^2 v^0}{\partial y\partial x}=0$$

$$\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)\frac{\partial^2 u^0}{\partial y^2}+\left(\nu+\frac{1-\nu}{2}\right)\frac{\partial^2 v^0}{\partial y\partial x}=0$$

$$\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)\frac{\partial^2 u^0}{\partial y^2}+\left(\frac{1+\nu}{2}\right)\frac{\partial^2 v^0}{\partial y\partial x}=0$$

$$\frac{\partial^2 u^0}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u^0}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v^0}{\partial y \partial x} = 0$$

$$\left(\frac{1+\nu}{2}\right)\frac{\partial^2 u^0}{\partial y \partial x} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v^0}{\partial x^2} + \frac{\partial^2 v^0}{\partial y^2} = 0$$

$$\frac{\partial^4 w^0}{\partial x^4} + 2\frac{\partial^4 w^0}{\partial x^2 \partial y^2} + \frac{\partial^4 w^0}{\partial y^4} = \frac{q}{D}$$

Right Edge

i. ***Either*** $\mathbf{N_x^+} = \mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{x}} + \nu \frac{\partial \mathbf{v}^0}{\partial \mathbf{y}}\right)$ ***or*** $\mathbf{u}^0$ ***must be specified.***

ii. ***Either*** $\mathbf{N_{xy}^+} = \left(\frac{1-\nu}{2}\right)\mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}^0}{\partial \mathbf{x}}\right)$ ***or*** $\mathbf{v}^0$ ***must be specified.***

iii. ***Either*** $\mathbf{Q_x^+} + \frac{\partial \mathbf{M_{xy}^+}}{\partial \mathbf{y}} = -\mathbf{D}\left(\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x}^3} + (2-\nu)\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x} \partial \mathbf{y}^2}\right)$

or $\mathbf{w}^0$ ***must be specified.***

iv. ***Either*** $\mathbf{M_x^+} = -\mathbf{D}\left(\frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{x}^2} - \nu \frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{y}^2}\right)$ ***or*** $\frac{\partial \mathbf{w}^0}{\partial \mathbf{x}}$ ***must be specified.***

Left Edge

i. *Either* $\mathbf{N}_x^- = \mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{x}} + \nu \frac{\partial \mathbf{v}^0}{\partial \mathbf{y}}\right)$ *or* $\mathbf{u}^0$ *must be specified.*

ii. *Either* $\mathbf{N}_{xy}^- = \left(\frac{1-\nu}{2}\right)\mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}^0}{\partial \mathbf{x}}\right)$ *or* $\mathbf{v}^0$ *must be specified.*

iii. *Either* $\mathbf{Q}_x^- + \frac{\partial \mathbf{M}_{xy}^-}{\partial \mathbf{y}} = -\mathbf{D}\left(\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x}^3} + (2-\nu)\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x}\partial \mathbf{y}^2}\right)$

or $\mathbf{w}^0$ *must be specified.*

iv. *Either* $\mathbf{M}_x^- = -\mathbf{D}\left(\frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{x}^2} - \nu \frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{y}^2}\right)$ *or* $\frac{\partial \mathbf{w}^0}{\partial \mathbf{x}}$ *must be specified.*

Front Edge

i. ***Either*** $\mathbf{N_y^+} = \mathbf{A}\left(\nu \frac{\partial \mathbf{u}^0}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}^0}{\partial \mathbf{y}}\right)$ ***or*** $\mathbf{u}^0$ ***must be specified.***

ii. ***Either*** $\mathbf{N_{yx}^+} = \left(\frac{1-\nu}{2}\right)\mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}^0}{\partial \mathbf{x}}\right)$ ***or*** $\mathbf{v}^0$ ***must be specified.***

iii. ***Either*** $\mathbf{Q_y^+} + \frac{\partial \mathbf{M_{yx}^+}}{\partial \mathbf{x}} = -\mathbf{D}\left(\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{y}^3} + (2-\nu)\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x}^2 \partial \mathbf{y}}\right)$

or $\mathbf{w}^0$ ***must be specified.***

iv. ***Either*** $\mathbf{M_y^+} = -\mathbf{D}\left(\nu \frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{x}^2} + \nu \frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{y}^2}\right)$ ***or*** $\frac{\partial \mathbf{w}^0}{\partial \mathbf{y}}$ ***must be specified.***

Back Edge

i. *Either* $\mathbf{N}_y^- = \mathbf{A}_{12}\frac{\partial \mathbf{u}^0}{\partial \mathbf{x}} + \mathbf{A}_{22}\frac{\partial \mathbf{v}^0}{\partial \mathbf{y}}$ *or* $\mathbf{u}^0$ *must be specified.*

ii. *Either* $\mathbf{N}_{yx}^- = \mathbf{A}_{66}\left(\frac{\partial \mathbf{u}^0}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}^0}{\partial \mathbf{x}}\right)$ *or* $\mathbf{v}^0$ *must be specified.*

iii. *Either* $\mathbf{Q}_y^- + \frac{\partial \mathbf{M}_{yx}^-}{\partial \mathbf{y}} = -\mathbf{D}_{22}\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{y}^3} - \left(\mathbf{D}_{12} + 4\mathbf{D}_{66}\right)\frac{\partial^3 \mathbf{w}^0}{\partial \mathbf{x}^2 \partial \mathbf{y}}$

or $\mathbf{w}^0$ *must be specified.*

iv. *Either* $\mathbf{M}_y^- = -\mathbf{D}_{11}\frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{x}^2} - \mathbf{D}_{12}\frac{\partial^2 \mathbf{w}^0}{\partial \mathbf{y}^2}$ *or* $\frac{\partial \mathbf{w}^0}{\partial \mathbf{x}}$ *must be specified.*

Limitations of the Classic Laminate Theory:

The classic laminate theory (CLT) is the simplest laminate theory one can get. While it has a reasonable range of applicability, it suffers a major deficiency associated with the transverse behaviour of laminates. A characteristic of laminated composites is their relative weakness in both stiffness and strength against transverse deformation as compared with conventional materials. Lower transverse shear stiffnesses would imply higher transverse shear strains under the same loads or stresses, undermining the Love-Kirchhoff's hypothesis. The lower transverse strength means that they are prone to transverse failure or, in other words, their failure is more sensitive to transverse stresses. The primitive way of evaluating transverse stresses is often insufficient to meet such a need. For better approximations, the classic laminate theory has to be improved, resulting in various types of more advanced laminate theories. While CLT possesses its unique form, advanced theories may range over a wide variety, depending on the factors taking into account for a specific requirement on accuracy.