Module 3

M3.Behaviour of Laminae - I

Learning Units of Module 3

M3.1 Stress-Strain Concepts in Three-Dimension

M3.2 Introduction to Anisotropic Elasticity

M3.3 Tensorial Concept and Indicial Notations

M3.4 Plane Stress Concept

Action of force (F) on a body

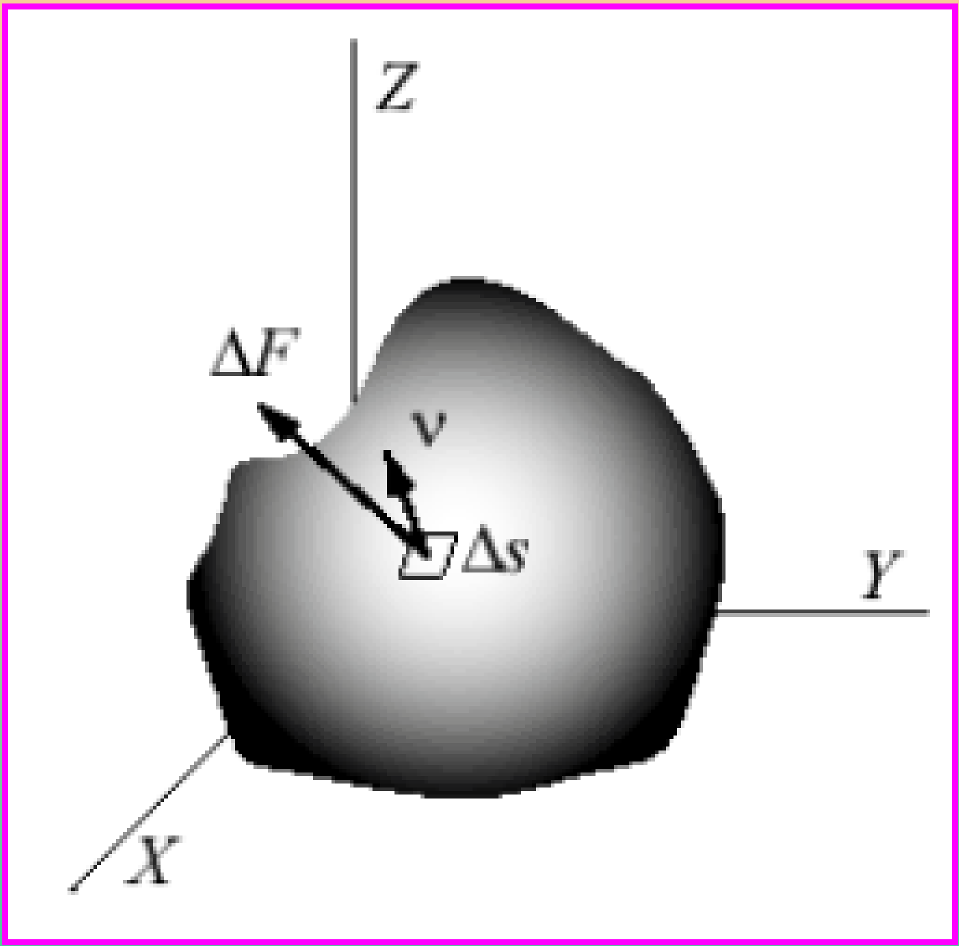


Figure: Action of force (F) on a body

Action of force (F) on a body

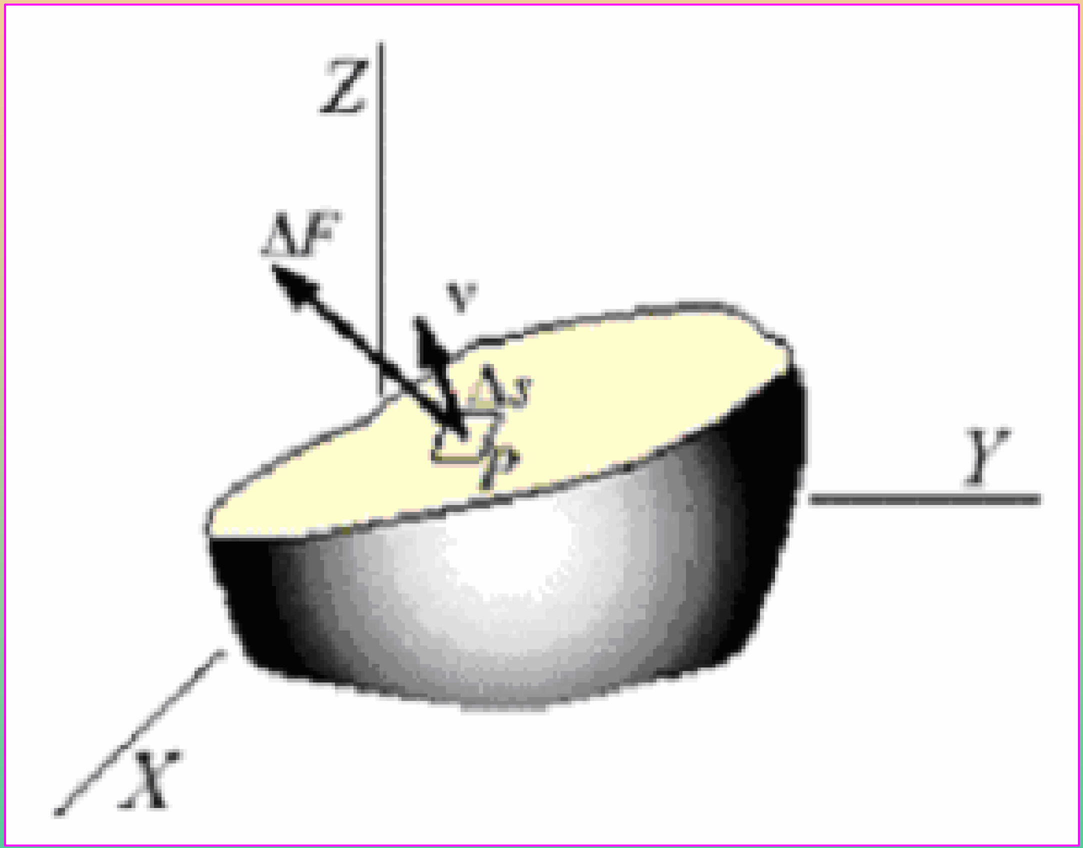


Figure: Stress at a Point

Action of force (F) on a body

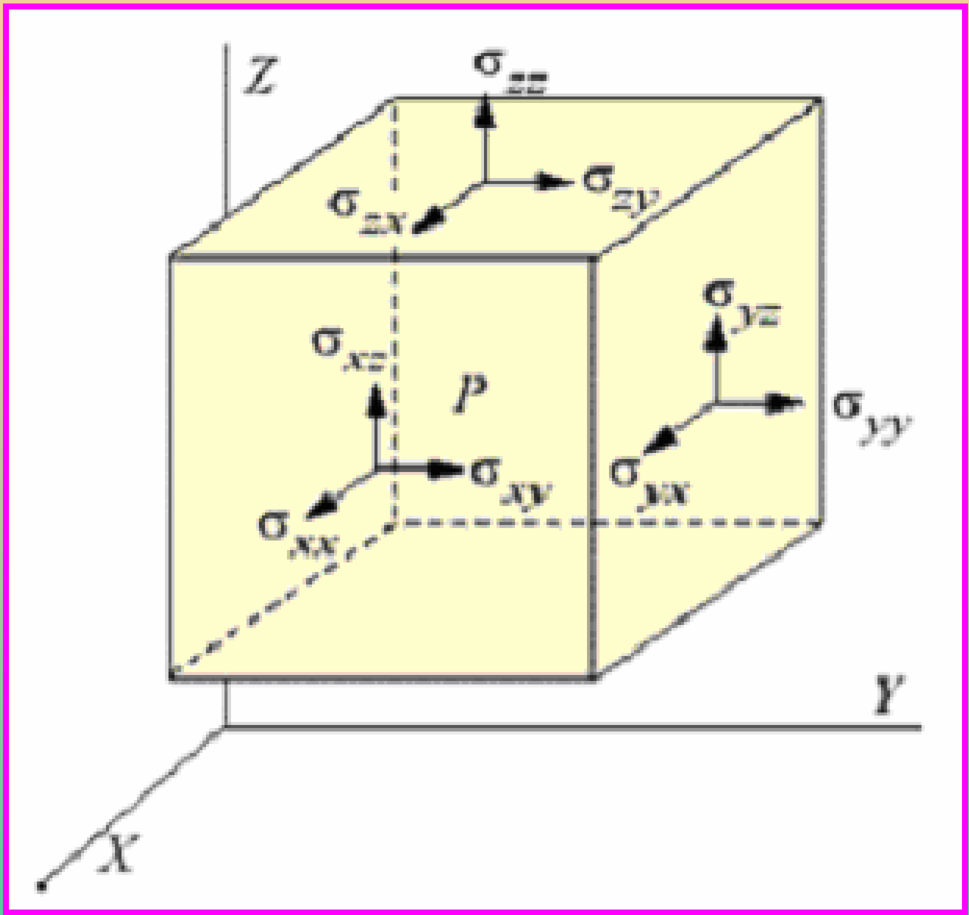


Figure: Elements 3-dimensional stress. All stresses have positive sense.

Global 1D Strain:

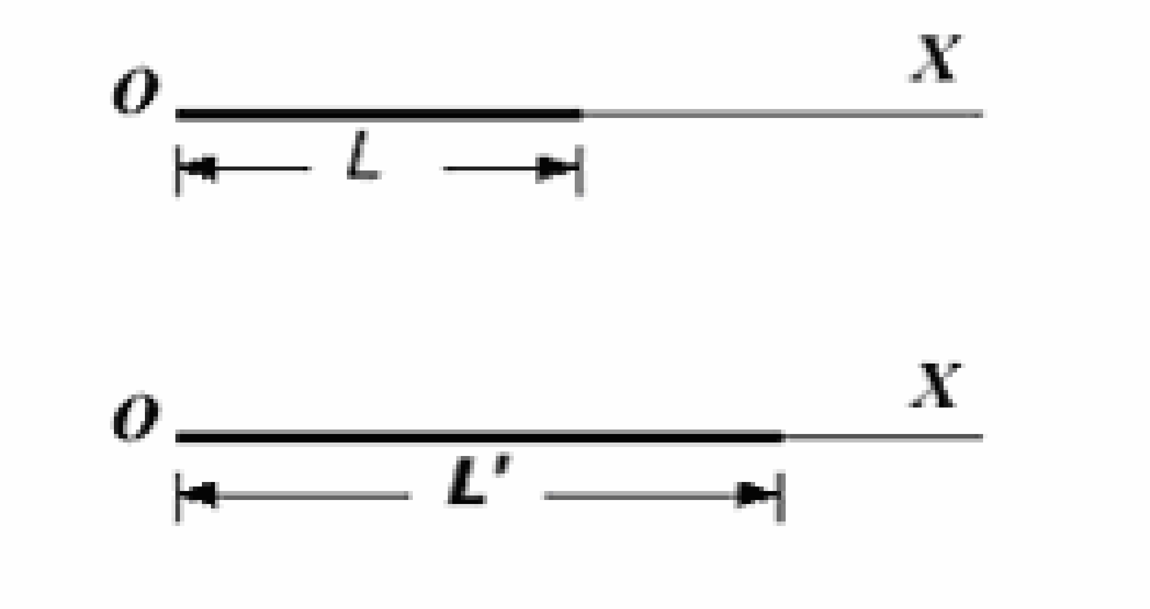


Figure: Global 1-dimensional strain.

Global 1D Strain:

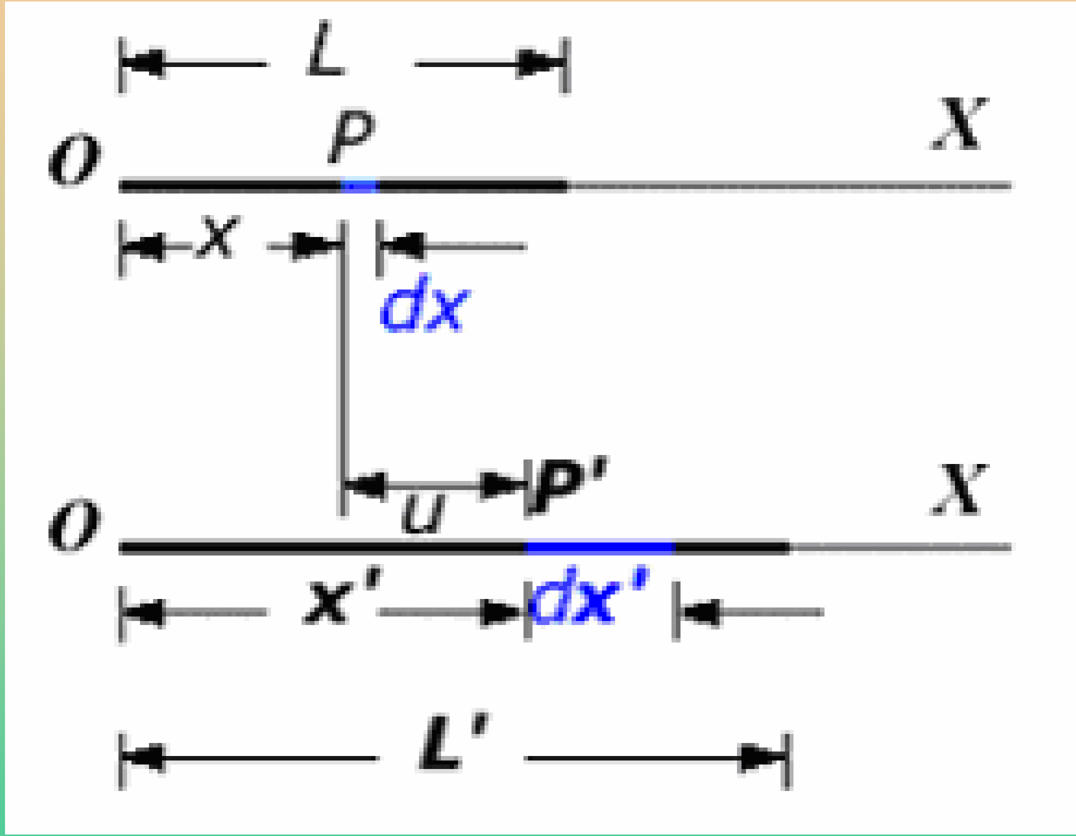


Figure: Infinitesimal 1-dimensional strain.

Global 3D Strain:

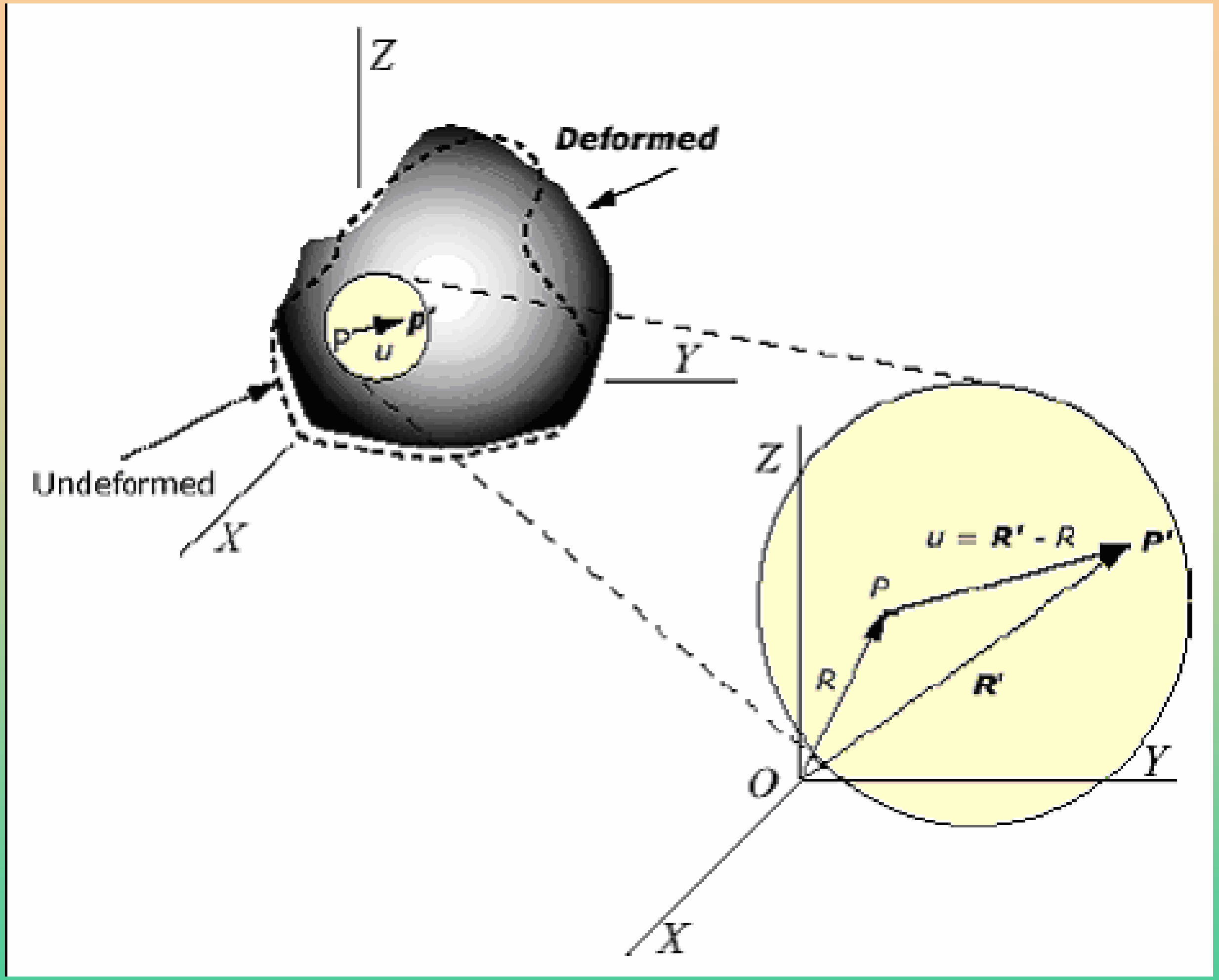


Figure M3.1.4: General definition of 3D-strain

Global 3D Strain:

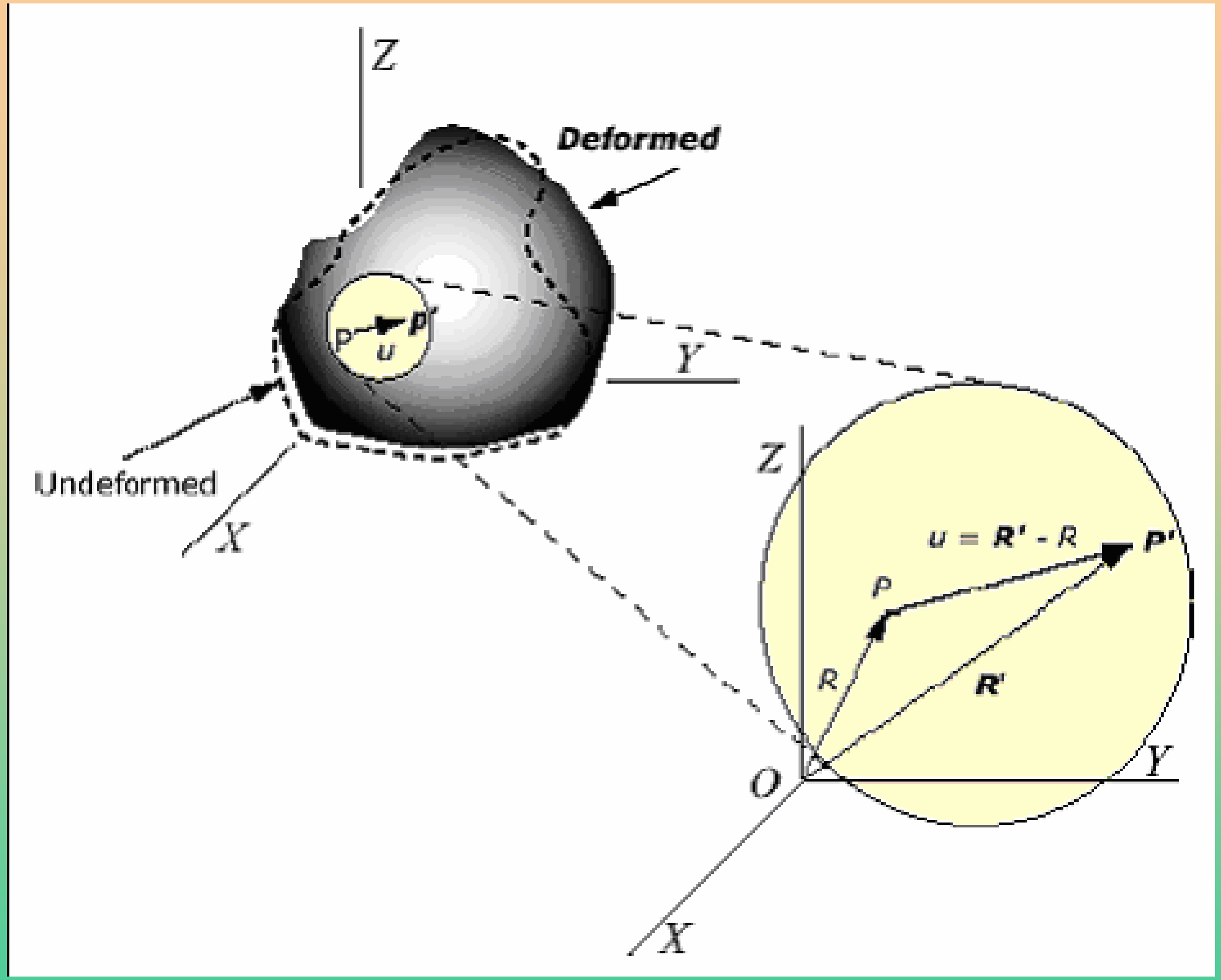


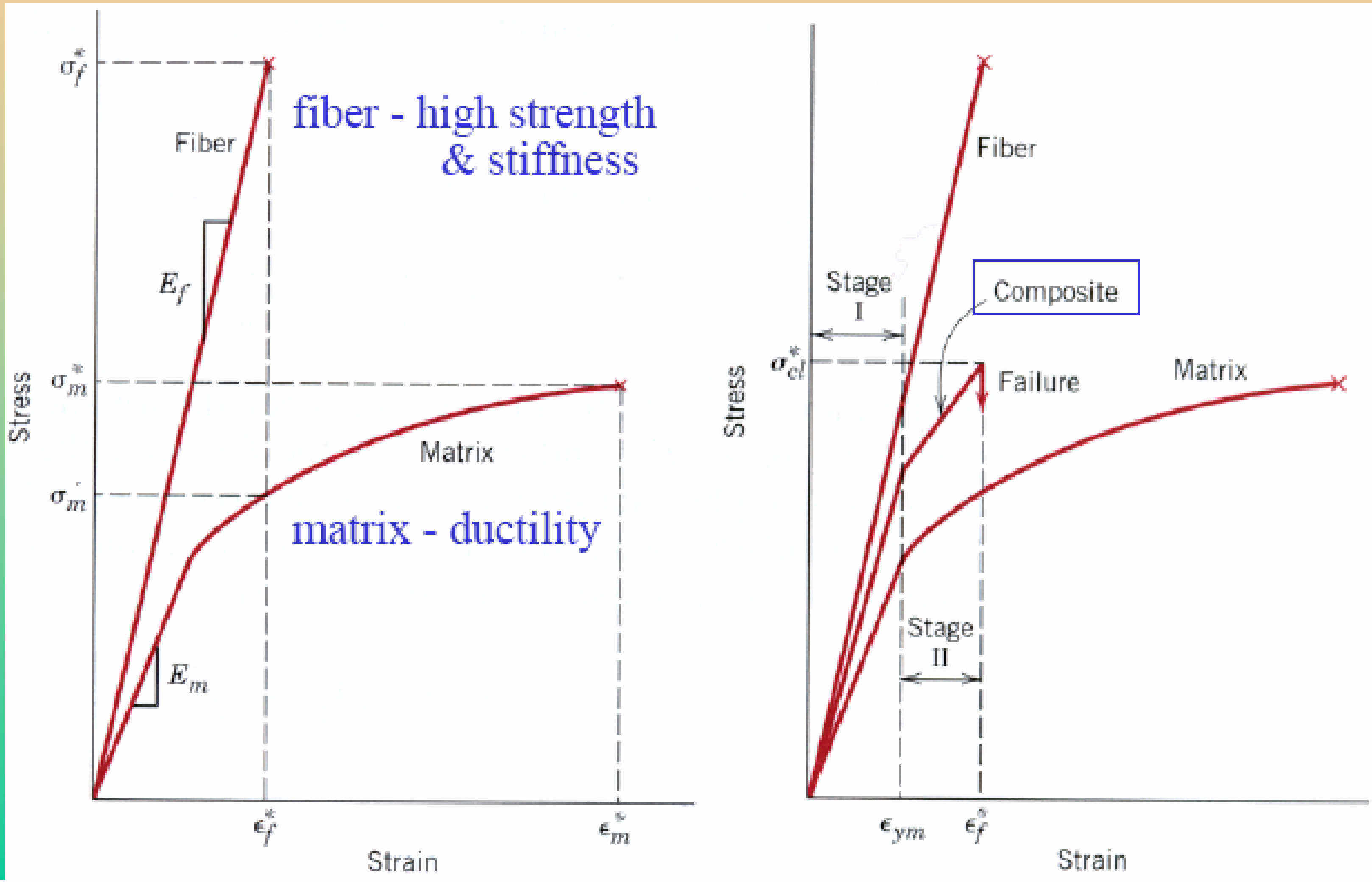
Figure M3.1.4: General definition of 3D-strain

Stress-Strain Curve of Composites

➢**Composite behavior**:

Matrix yields, fibers take load

Failure at, ε_f but not catastrophic



Lamina Stress-Strain Relationships

Stress Components

9 components of stress:

$$\sigma_{ij} \qquad i, j = 1, 2, 3$$

Types of Stress

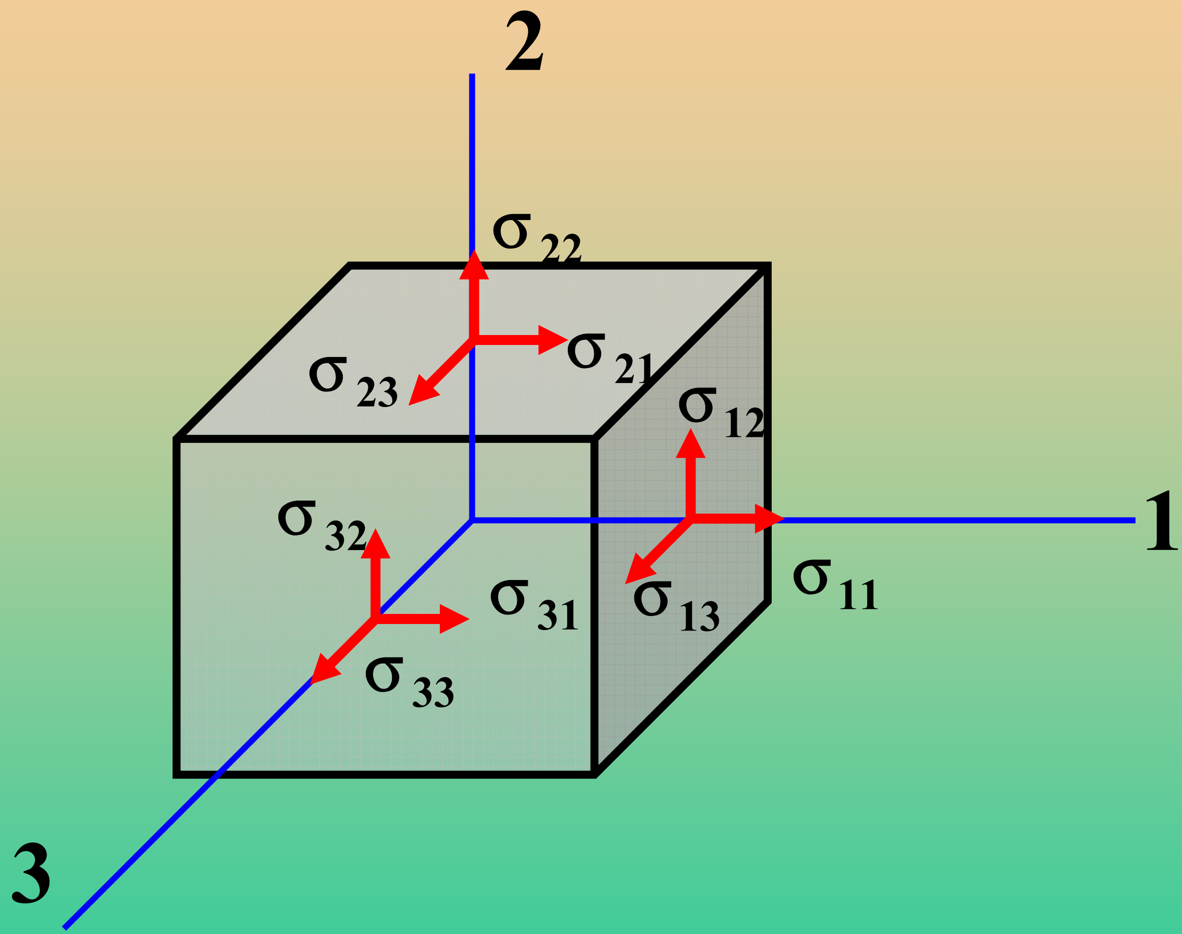
1. Normal Stress $\mathbf{i = j}$

2. Shear Stress $\mathbf{i \neq j}$

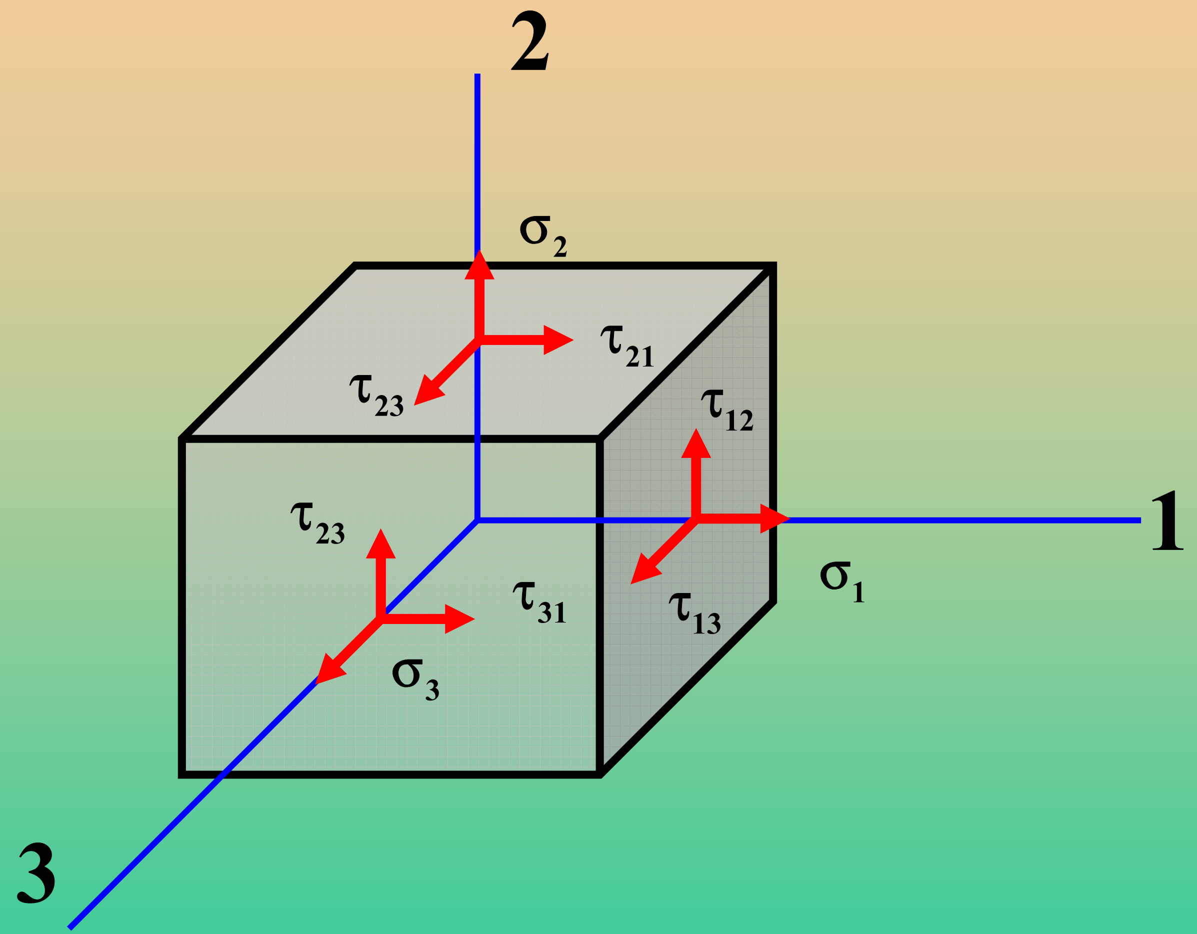
Subscripts

- First subscript refers to direction of outer normal
- Second subscript refers to the direction in which the stress acts

Stress Cube



Stress Cube



Strain

- ✓ **Corresponding to each stress component, there is a strain component, ε_{ij} describing the deformation at a point.**
- ✓ **Normal strains describe the extension per unit length.**
- ✓ **Shear strains describe distortional deformation.**
- ✓ **Tensor and Engineering Shear Strains**

STRESSES		STRAINS	
Tensor Notation	Contracted Notation	Tensor Notation	Contracted Notation
σ_{11} (σ_1)	σ_1	ε_{11} (ε_1)	ε_1
σ_{22} (σ_2)	σ_2	ε_{22} (ε_2)	ε_2
σ_{33} (σ_3)	σ_3	ε_{33} (ε_3)	ε_3
τ_{23} (σ_{23})	σ_4	$\gamma_{23} = 2\varepsilon_{23}$	ε_4
τ_{31} (σ_{31})	σ_5	$\gamma_{31} = 2\varepsilon_{31}$	ε_5
τ_{12} (σ_{12})	σ_6	$\gamma_{12} = 2\varepsilon_{12}$	ε_6

General Formula

Stresses and strains are related to each other. The most general form of this relationship is:

$$\sigma_{ij} = f_{ij}\left(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{32}, \varepsilon_{33}\right)$$

Linear Elastic Material

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\
C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\
C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & C_{3232} & C_{3213} & C_{3221} \\
C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & C_{1332} & C_{1313} & C_{1321} \\
C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{Bmatrix}
$$

9 Stresses x 9 Strains =
81 Components in relationship

Symmetry

$$\sigma_{ij} = \sigma_{ji} \qquad i \neq j$$

and

$$\varepsilon_{ij} = \varepsilon_{ji} \qquad i \neq j$$

Symmetry

$$C_{ijkl} = C_{jikl}$$

and

$$C_{ijkl} = C_{ijlk}$$

6 Stresses x 6 Strains =
36 Components in relationship

Contracted Notation: Stresses

$$\sigma_{11} = \sigma_1$$

$$\sigma_{22} = \sigma_2$$

$$\sigma_{33} = \sigma_3$$

$$\sigma_{23} = \sigma_{32} = \sigma_4$$

$$\sigma_{31} = \sigma_{13} = \sigma_5$$

$$\sigma_{12} = \sigma_{21} = \sigma_6$$

Contracted Notation : Strains

$$\varepsilon_{11} = \varepsilon_1$$

$$\varepsilon_{22} = \varepsilon_2$$

$$\varepsilon_{33} = \varepsilon_3$$

$$2\varepsilon_{23} = 2\varepsilon_{32} = \gamma_{23} = \gamma_{32} = \varepsilon_4$$

$$2\varepsilon_{13} = 2\varepsilon_{31} = \gamma_{13} = \gamma_{31} = \varepsilon_5$$

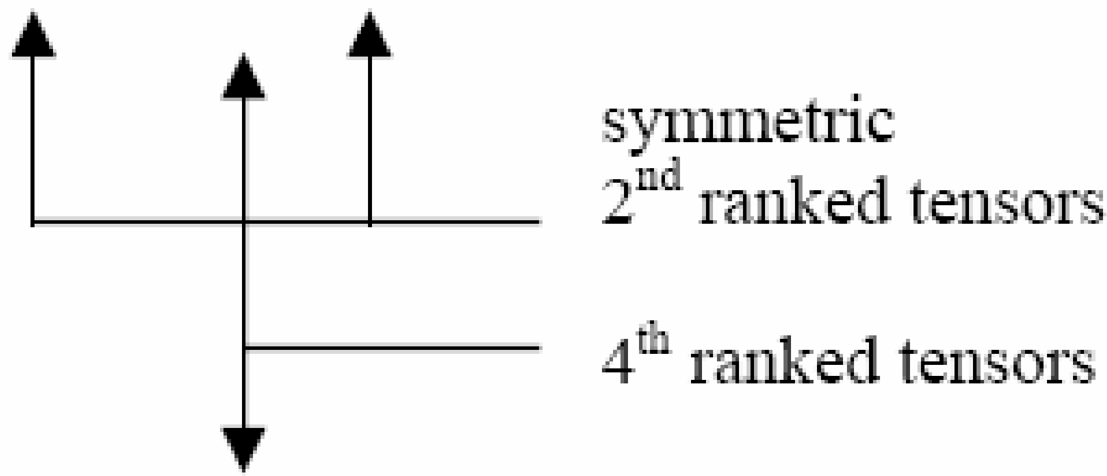
$$2\varepsilon_{12} = 2\varepsilon_{21} = \gamma_{12} = \gamma_{21} = \varepsilon_6$$

Hooke's Law (Stiffness)

- S is the inverse of C. These equations encompass all anisotropic crystalline behavior.

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

C is the stiffness tensor



Similarly, $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$

S is the compliance tensor

Hooke's Law (Stiffness)

$$\sigma_i = C_{ij}\varepsilon_j \quad i, j = 1, 2, \ldots, 6$$

or

$$\{\sigma\} = [C]\{\varepsilon\}$$

Hooke's Law

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Hooke's Law (Compliance)

$$\varepsilon_i = S_{ij}\sigma_j \qquad i, j = 1, 2, \ldots, 6$$

or

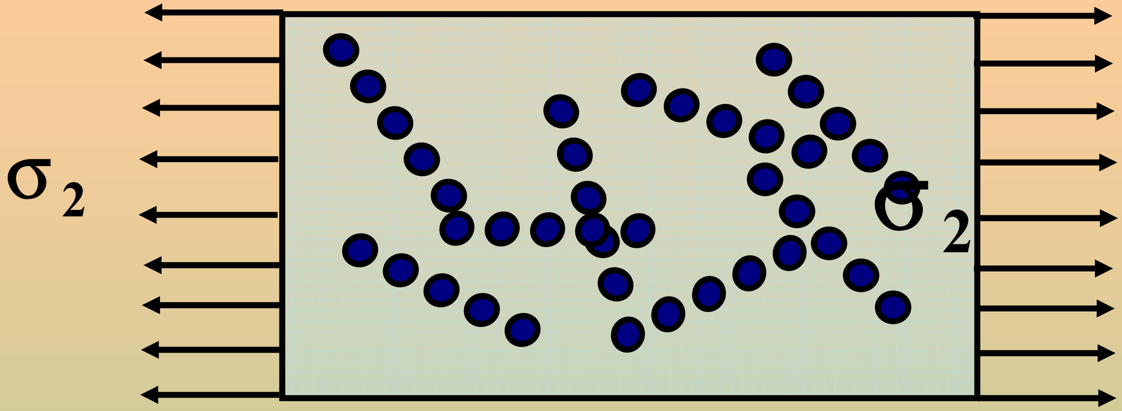
$$\{\varepsilon\} = [S]\{\sigma\}$$

Inverse Relationship

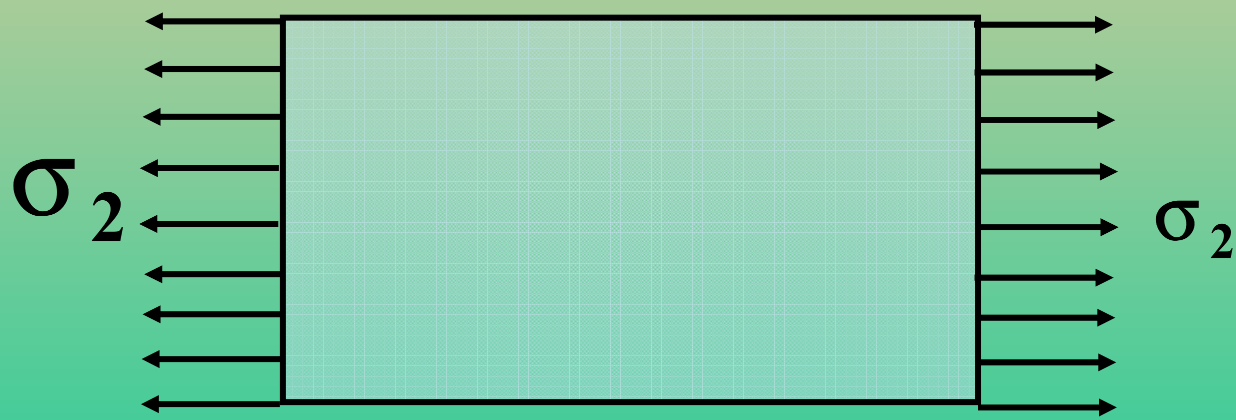
$$[S] = [C]^{-1}$$

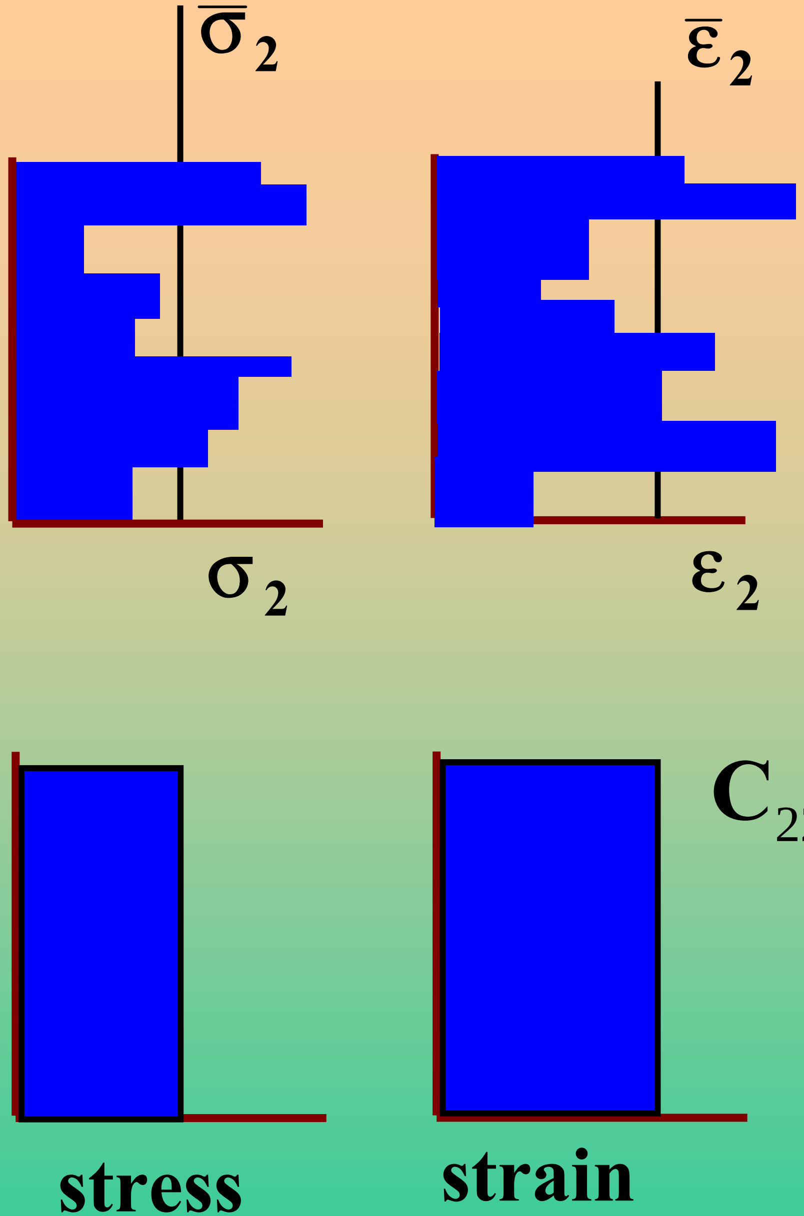
[S] and [C] are symmetric matrices!

Heterogeneous Composite



Equivalent Homogeneous Composite





$$\bar{\sigma}_2 = C_{22}\bar{\varepsilon}_{22}$$

$C_{22} \rightarrow$ ***effective modulus.***

Average Stresses and Strains

$$\bar{\sigma}_i = \frac{\int_V \sigma_i dV}{\int_V dV} = \frac{\int_V \sigma_i dV}{V}$$

$$i = 1, 2, \ldots, 6$$

$$\bar{\varepsilon}_i = \frac{\int_V \varepsilon_i dV}{\int_V dV} = \frac{\int_V \varepsilon_i dV}{V}$$

Average Values

$$\{\bar{\sigma}\} = [C]\{\bar{\varepsilon}\}$$

and

$$\{\bar{\varepsilon}\} = [S]\{\bar{\sigma}\}$$

We use the effective (or average) values of stress, strain and moduli when referring to lamina behavior.

Strain Energy Density

$$W = \frac{1}{2} C_{ij} \varepsilon_i \varepsilon_j$$

and

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} = C_{ij} \varepsilon_j$$

Second Derivatives

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij}$$

and

$$\frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = C_{ji}$$

Symmetry

$$C_{ij} = C_{ji}$$

and

$$S_{ij} = S_{ji}$$

Symmetry: 21 Components

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\ M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\ M & \mathbf{SYM} & \Lambda & \Lambda & C_{55} & C_{56} \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

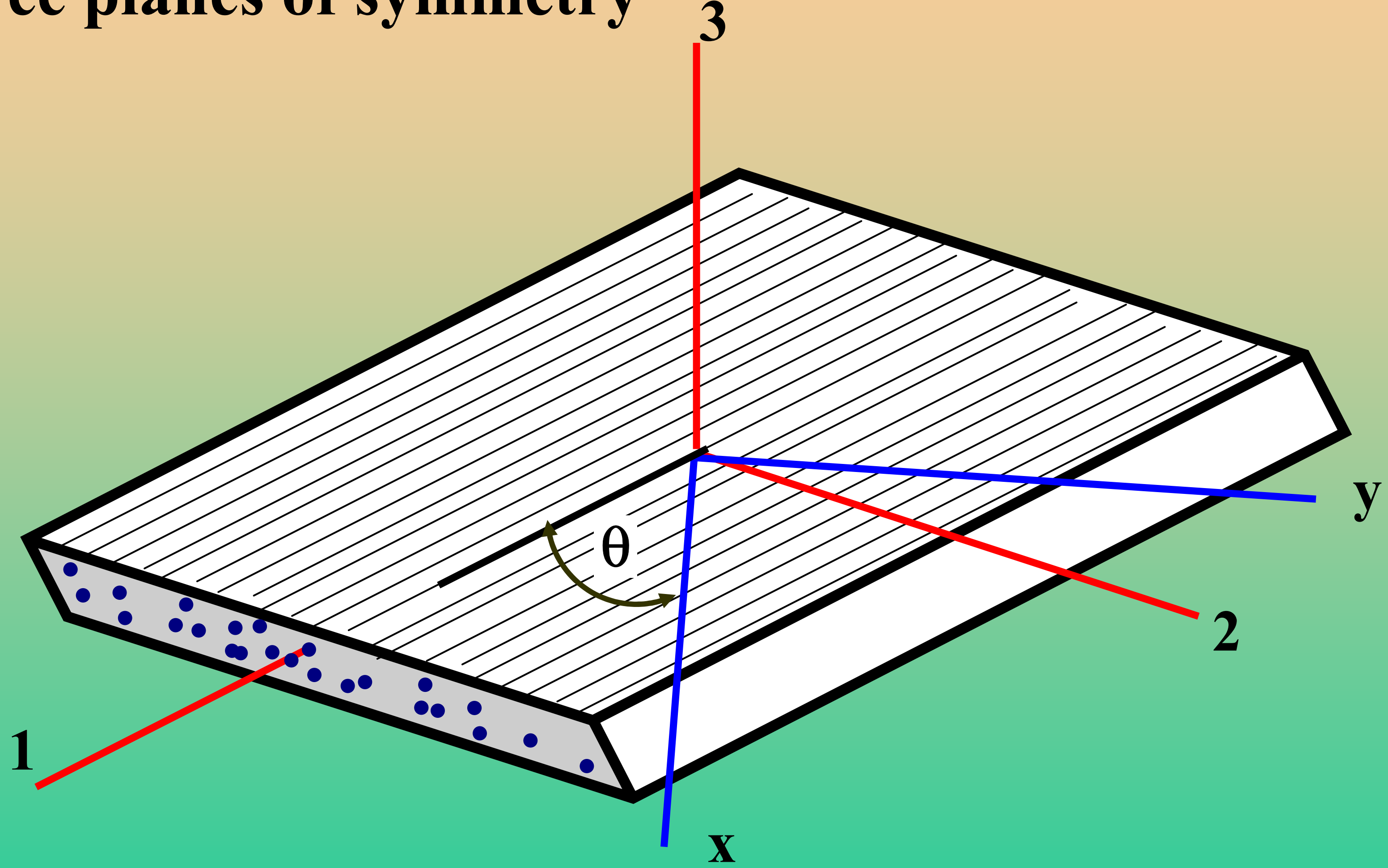
Monoclinic Material; 13 Components

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
M & C_{22} & C_{23} & 0 & 0 & C_{26} \\
M & \Lambda & C_{33} & 0 & 0 & C_{36} \\
M & \Lambda & \Lambda & C_{44} & C_{45} & 0 \\
M & \mathrm{SYM} & \Lambda & \Lambda & C_{55} & 0 \\
M & \Lambda & \Lambda & 0 & 0 & C_{66}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

1 plane of symmetry

Orthotropic Material: 9 Constants

Three planes of symmetry



Specially Orthotropic Material

Three planes of symmetry 1-2-3 Directions are principal coordinate directions corresponding to symmetry planes, as shown on previous slide.

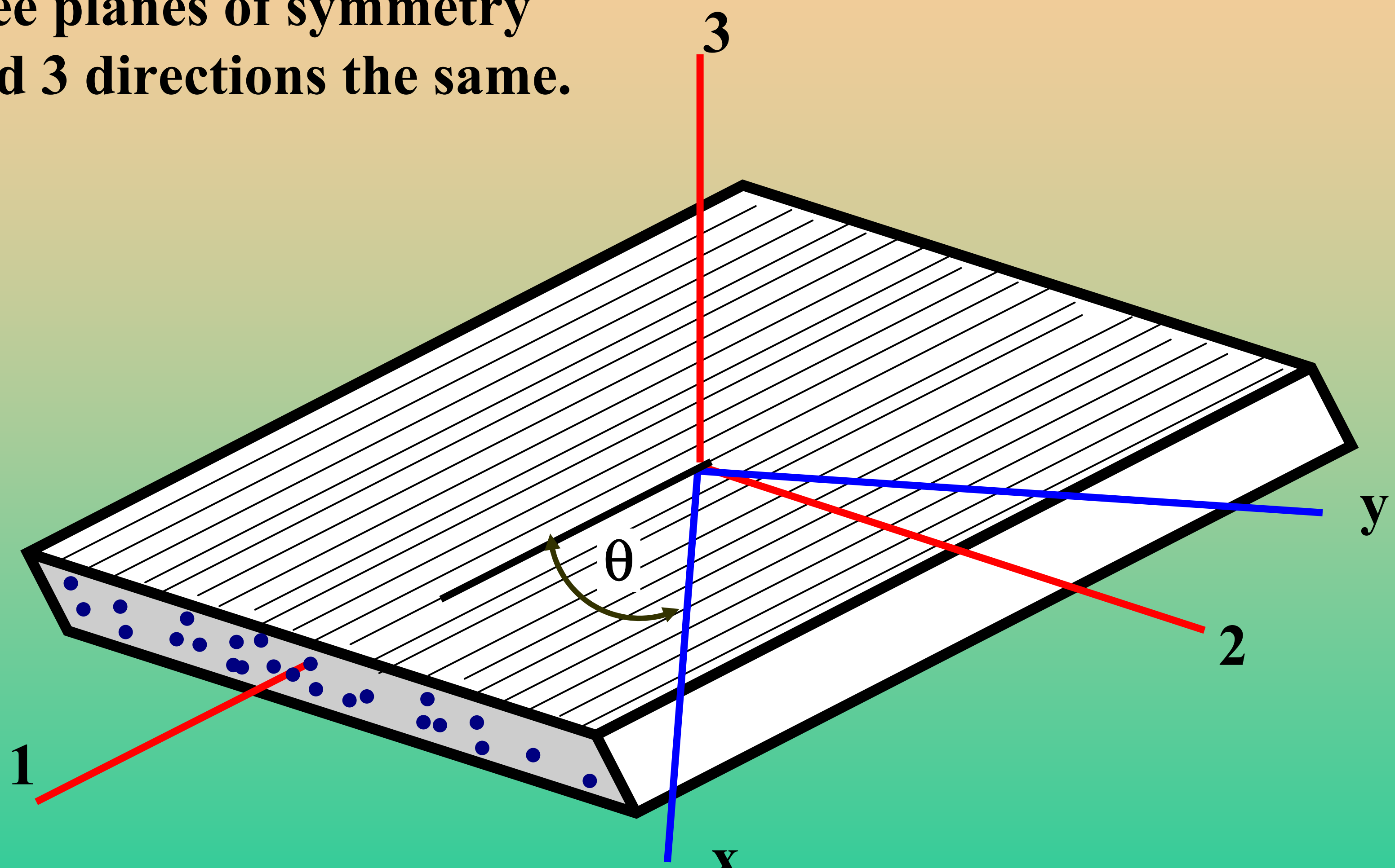
Orthotropic Stiffness Matrix

9 Components

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ M & C_{22} & C_{23} & 0 & 0 & 0 \\ M & \Lambda & C_{33} & 0 & 0 & 0 \\ M & \Lambda & \Lambda & C_{44} & 0 & 0 \\ M & \mathbf{SYM} & \Lambda & \Lambda & C_{55} & 0 \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

Transversely Isotropic Material

Three planes of symmetry
2 and 3 directions the same.



Stiffness Matrix

5 Components

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ M & C_{22} & C_{23} & 0 & 0 & 0 \\ M & \Lambda & C_{22} & 0 & 0 & 0 \\ M & \Lambda & \Lambda & \frac{(C_{22}-C_{23})}{2} & 0 & 0 \\ M & \mathrm{SYM} & \Lambda & \Lambda & C_{66} & 0 \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Isotropic Material

Three planes of symmetry
1,2 and 3 directions the same.

Stiffness Matrix

2 Components

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
M & C_{11} & C_{12} & 0 & 0 & 0 \\
M & \Lambda & C_{11} & 0 & 0 & 0 \\
M & \Lambda & \Lambda & \frac{(C_{11}-C_{12})}{2} & 0 & 0 \\
M & SYM\ \Lambda & & \Lambda & \frac{(C_{11}-C_{12})}{2} & 0 \\
M & \Lambda & \Lambda & \Lambda & \Lambda & \frac{(C_{11}-C_{12})}{2}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

Anisotropic Material

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\ M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\ M & \mathbf{SYM} & \Lambda & \Lambda & C_{55} & C_{56} \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

Anisotropic Material

Extension

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\
M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\
M & \mathbf{SYM} & \Lambda & \Lambda & C_{55} & C_{56} \\
M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

Anisotropic Material

Extension-Extension Coupling

Extension

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\ M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\ M & \mathrm{SYM} & \Lambda & \Lambda & C_{55} & C_{56} \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Extension-Extension Coupling

Shear-Extension Coupling

Extension

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\
M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\
M & SYM & \Lambda & \Lambda & C_{55} & C_{56} \\
M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
$$

Extension-Extension Coupling

Shear-Extension Coupling

Extension

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\ M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\ M & SYM & \Lambda & \Lambda & C_{55} & C_{56} \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Shear

Extension-Extension Coupling

Shear-Extension Coupling

Extension

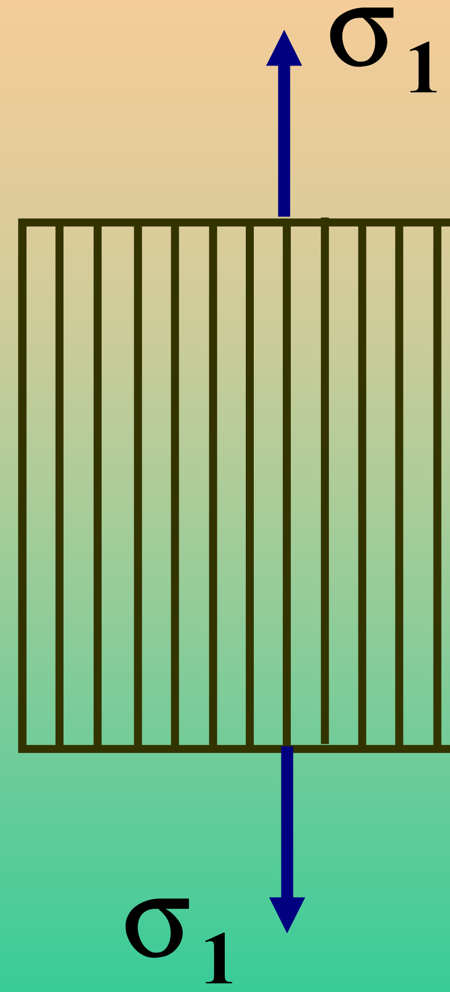
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ M & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ M & \Lambda & C_{33} & C_{34} & C_{35} & C_{36} \\ M & \Lambda & \Lambda & C_{44} & C_{45} & C_{46} \\ M & \mathrm{SYM} & \Lambda & \Lambda & C_{55} & C_{56} \\ M & \Lambda & \Lambda & \Lambda & \Lambda & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Shear

Shear-Shear Coupling

Material	Nonzero Terms	Independent terms
3D		
Anisotropic	36	21
Generally Orthotropic	36	9
Specially Orthotropic	12	9
Transversely Isotropic	12	5
Isotropic	12	2
2D		
Anisotropic	9	6
Generally Orthotropic	9	4
Specially Orthotropic	5	4
Balanced Orthotropic	5	3
Isotropic	5	2

Uniaxial Load in Fiber Direction



Resulting Strains

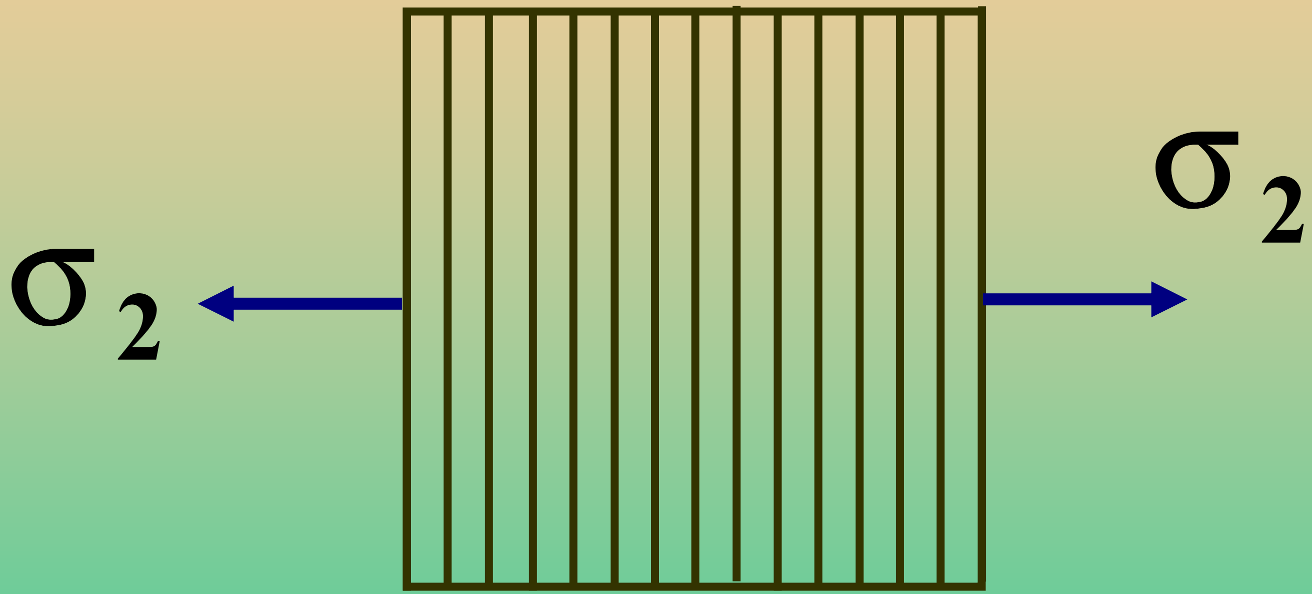
$$\varepsilon_1 = \frac{\sigma_1}{\mathbf{E}_1}$$

$$\varepsilon_2 = -\nu_{12}\varepsilon_1 = -\nu_{12}\frac{\sigma_1}{\mathbf{E}_1}$$

$$\varepsilon_3 = -\nu_{13}\varepsilon_1 = -\nu_{13}\frac{\sigma_1}{\mathbf{E}_1}$$

$$\gamma_{12} = \gamma_{23} = \gamma_{13} = 0 \quad \textbf{\textit{or}} \quad \varepsilon_6 = \varepsilon_4 = \varepsilon_5 = 0$$

Transverse Load



Resulting Strains

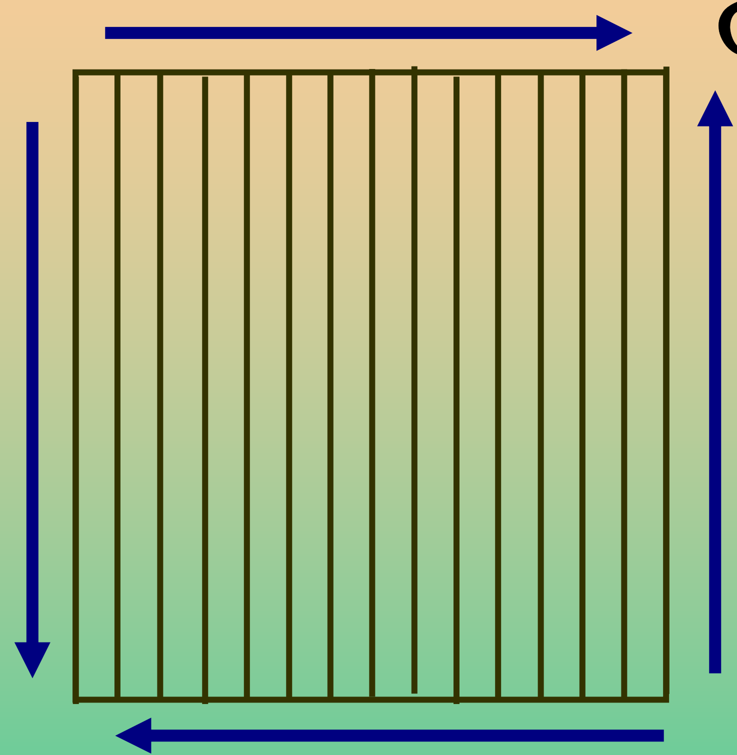
$$\varepsilon_2 = \frac{\sigma_2}{E_2}$$

$$\varepsilon_1 = -\nu_{21}\varepsilon_2 = -\nu_{21}\frac{\sigma_2}{E_2}$$

$$\varepsilon_3 = -\nu_{31}\varepsilon_2 = -\nu_{31}\frac{\sigma_2}{E_2}$$

$$\gamma_{12} = \gamma_{23} = \gamma_{13} = 0 \quad \text{or} \quad \varepsilon_6 = \varepsilon_4 = \varepsilon_5 = 0$$

Shear Load



$\sigma_{12} = \sigma_6$

$\sigma_{12} = \sigma_6$

Resulting Strains

$$\gamma_{12} = \varepsilon_6 = \frac{\sigma_6}{G_{12}}$$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_{23} = \gamma_{13} = 0$$

or

$$\varepsilon_4 = \varepsilon_5 = 0$$

Engineering Material Properties

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\mathbf{E}_1} & -\frac{\nu_{21}}{\mathbf{E}_2} & -\frac{\nu_{31}}{\mathbf{E}_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{\mathbf{E}_1} & \frac{1}{\mathbf{E}_2} & -\frac{\nu_{32}}{\mathbf{E}_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{\mathbf{E}_1} & -\frac{\nu_{23}}{\mathbf{E}_2} & \frac{1}{\mathbf{E}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mathbf{G}_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mathbf{G}_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mathbf{G}_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix}$$

Engineering Material Properties

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Engineering Material Properties

$E_1, E_2, E_3 =$ *Young's moduli in* 1-, 2- *and* 3- *directions*

$\nu_{12}, \nu_{13}, \nu_{23} =$ *Poisson's ratios (extension-extension coupling)*

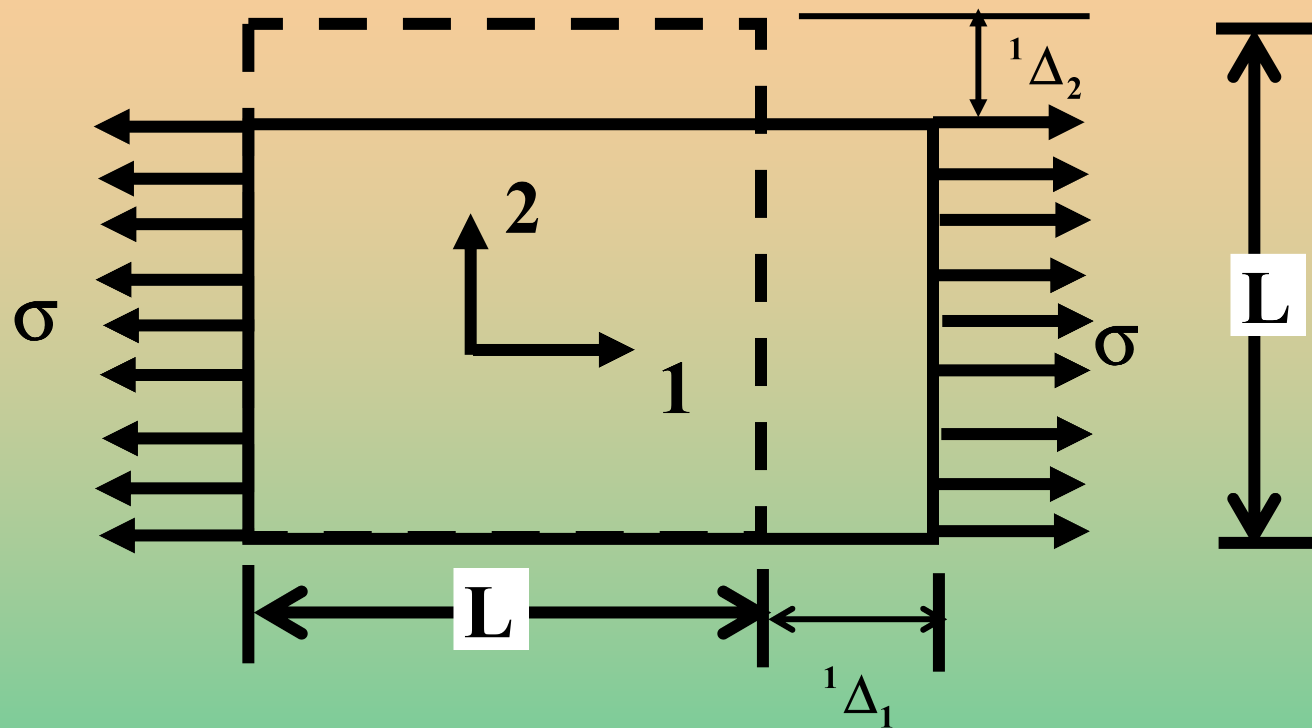
$G_{23}, G_{31}, G_{12} =$ *Shear moduli in* 2-3, 3-1, *and* 1-2 *directions*

Symmetry

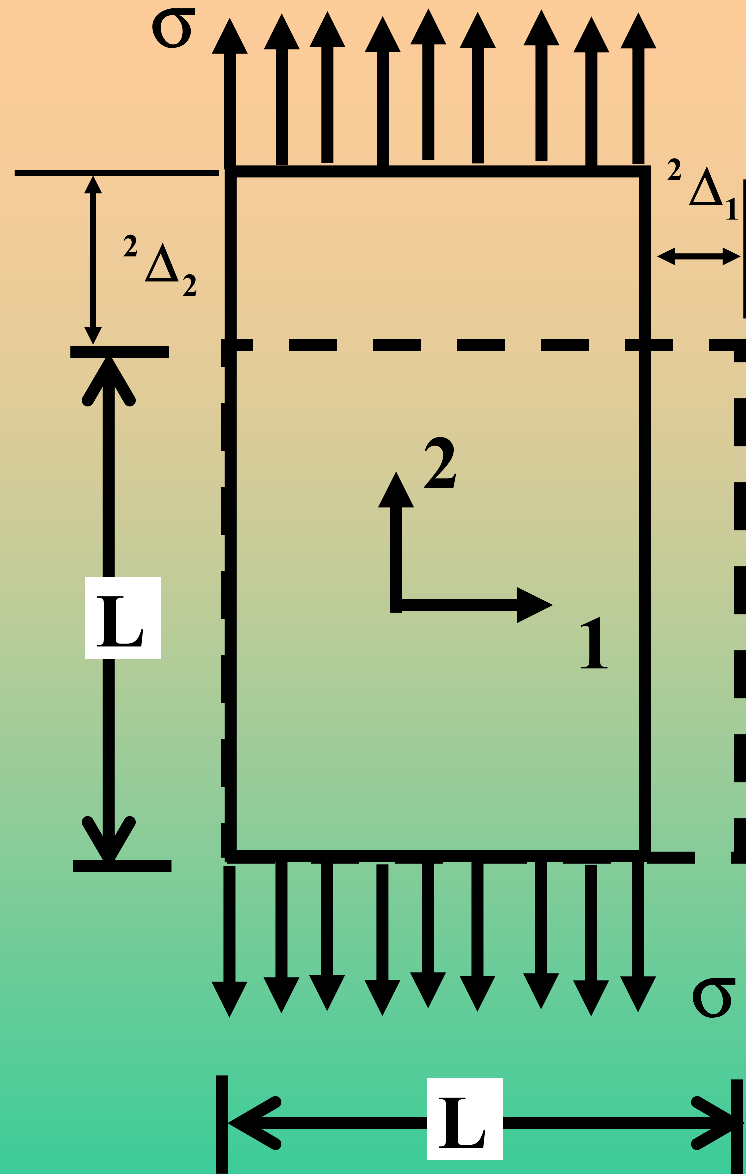
$$[S] = [C]^{-1}$$

Due to Symmetry :

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$$



2
1
σ
σ
L
L
$^1\Delta_2$
$^1\Delta_1$


σ
2Δ1
2Δ2
2
1
L
σ
L

$^{1}\varepsilon_1 = \frac{\sigma}{E_1}$	$^{1}\varepsilon_2 = -\frac{\nu_{12}\sigma}{E_1}$
$^{1}\Delta_1 = \frac{\sigma L}{E_1}$	$^{1}\Delta_2 = -\frac{\nu_{12}\sigma L}{E_1}$

$^{2}\varepsilon_1 = -\frac{\nu_{21}\sigma}{E_2}$	$^{2}\varepsilon_2 = \frac{\sigma}{E_2}$
$^{2}\Delta_1 = -\frac{\nu_{21}\sigma L}{E_2}$	$^{2}\Delta_2 = \frac{\sigma L}{E_2}$

$$^{1}\Delta_{2} = {}^{2}\Delta_{1}$$

$$-\frac{\nu_{12}\sigma L}{E_1} = -\frac{\nu_{21}\sigma L}{E_2}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

Inverse Relationship

$$[S] = [C]^{-1}$$

[S] and [C] are symmetric matrices!

Inverse Relationship

$$C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{S} \qquad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S} \qquad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}$$

$$C_{22} = \frac{S_{11}S_{33} - S_{13}^2}{S} \qquad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S} \qquad C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{S}$$

$$C_{44} = \frac{1}{S_{44}} \qquad C_{55} = \frac{1}{S_{55}} \qquad C_{66} = \frac{1}{S_{66}}$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}$$

Inverse Relationship

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} \quad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta}$$

$$C_{44} = G_{23} \quad C_{55} = G_{31} \quad C_{66} = G_{12}$$

$$C_{12} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta}$$

$$C_{13} = \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta}$$

$$C_{23} = \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2\Delta}$$

$$\Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

Isotropic Material

$$E_1 = E_2 = E_3 = E$$

$$G = \frac{E}{2(1+\nu)}$$

$$-1 < \nu < 0.5$$

Orthotropic Material Constraints

$$S_{11}, S_{22}, S_{33}, S_{44}, S_{55}, S_{66} > 0$$

$$\Downarrow$$

$$E_1, E_2, E_3, G_{23}, G_{31}, G_{12} > 0$$

Orthotropic Material Constraints

$$C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66} > 0$$

$$\Downarrow$$

$$(1-\nu_{23}\nu_{32}), (1-\nu_{13}\nu_{31}), (1-\nu_{12}\nu_{21}) > 0$$

$$\Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3} > 0$$

Orthotropic Material Constraints

$$|S_{23}| < \sqrt{S_{22}S_{33}}$$

$$|S_{13}| < \sqrt{S_{11}S_{33}}$$

$$|S_{12}| < \sqrt{S_{11}S_{22}}$$

Orthotropic Material Constraints

$$(1-\nu_{23}\nu_{32}),(1-\nu_{13}\nu_{31}),(1-\nu_{12}\nu_{21}) > 0$$

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad i,j = 1,2,3$$

$$|\nu_{21}| < \sqrt{\frac{E_2}{E_1}} \qquad |\nu_{32}| < \sqrt{\frac{E_3}{E_2}} \qquad |\nu_{13}| < \sqrt{\frac{E_1}{E_3}}$$

$$|\nu_{12}| < \sqrt{\frac{E_1}{E_2}} \qquad |\nu_{23}| < \sqrt{\frac{E_2}{E_3}} \qquad |\nu_{13}| < \sqrt{\frac{E_3}{E_1}}$$

Plane Stress
Orthotropic Material

$$\sigma_3 = \sigma_4 = \sigma_5 = 0$$

or

$$\sigma_3 = \tau_{23} = \tau_{31} = 0$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}$$

Plane Stress
Orthotropic Material

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

Plane Stress
Orthotropic Material

$$\begin{Bmatrix} \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{Bmatrix} S_{13}\sigma_1 + S_{23}\sigma_2 \\ 0 \\ 0 \end{Bmatrix}$$

Compliance

Compliances :

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{12} = S_{21} = -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1}$$

$$S_{66} = \frac{1}{G_{12}}$$

Compliance

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{12}}{E_1} & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Compliance

$$\{\varepsilon\} = [S]\{\sigma\}$$

Compliances:

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2} \qquad S^{*}_{44} = \frac{1}{G_{23}}$$

$$S_{12} = S_{21} = -\frac{\nu_{12}}{E_1} \qquad S^{*}_{55} = \frac{1}{G_{13}}$$

$$S_{66} = \frac{1}{G_{12}}$$

Stiffness

$$\{\sigma\} = [\mathbf{Q}]\{\varepsilon\}$$

Lamina Stiffness Matrix

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Stiffness Terms

$$Q_{11} = \frac{S_{22}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{E_1}{\left(1 - \nu_{12}\nu_{21}\right)}$$

$$Q_{22} = \frac{S_{11}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{E_2}{\left(1 - \nu_{12}\nu_{21}\right)}$$

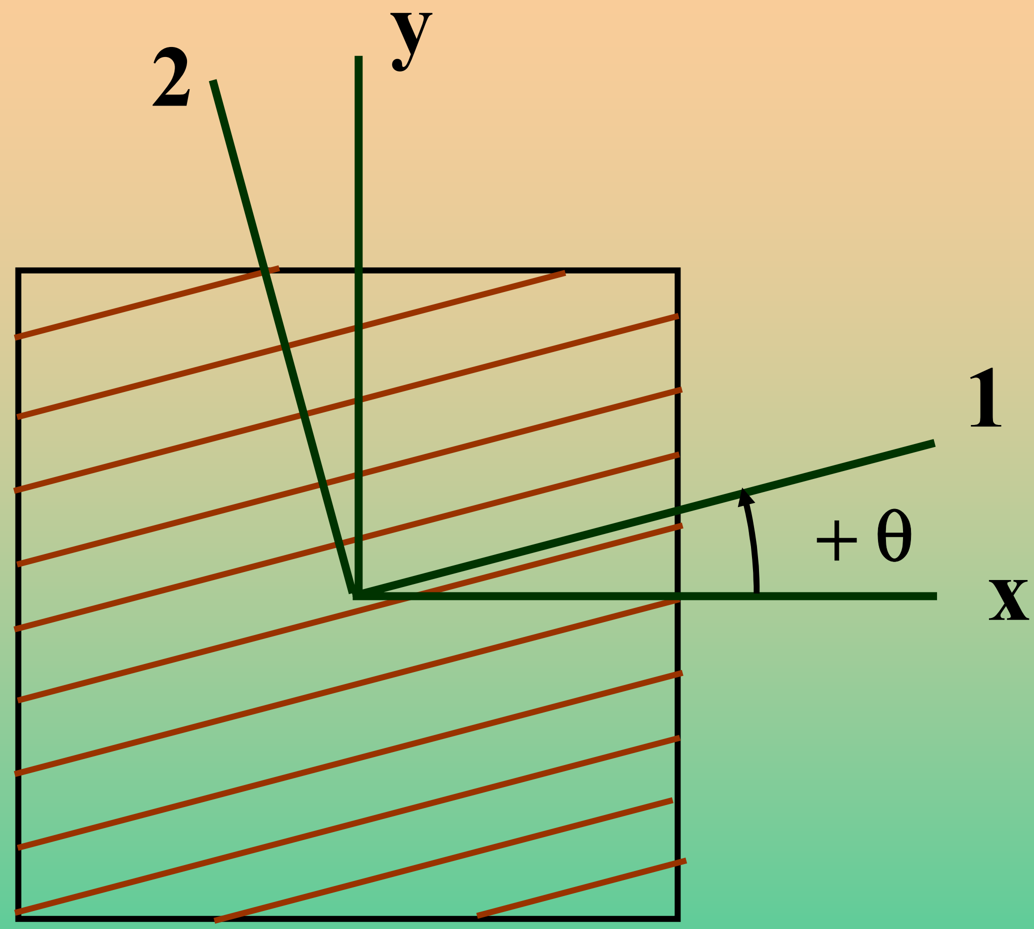
$$Q_{12} = Q_{21} = -\frac{S_{12}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{\nu_{12}E_2}{\left(1 - \nu_{12}\nu_{21}\right)}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

Some Typical Properties

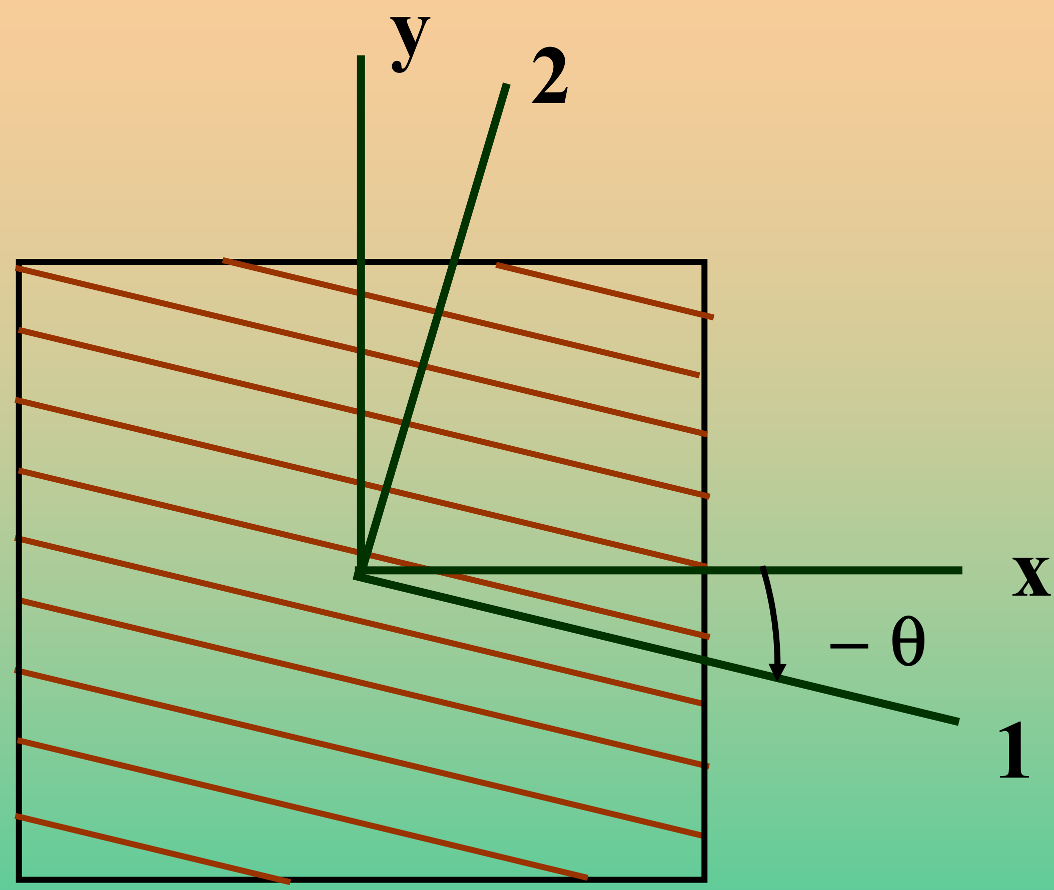
Material	E_1 (Msi)	E_2 (Msi)	G_{12} (Msi)	n_{12}
T300/934 Graphite /Epoxy	19.0	1.5	1.0	0.22
AS/3501 Graphite /Epoxy	20.0	1.3	1.0	0.3
p-100/ERL 1962 Pitch Graphite /Epoxy	68.0	0.9	0.81	0.31
Kevlar® 49 /934 Aramid/Epoxy	11.0	0.8	0.33	0.34
Scotchply® 1002 E-glass/Epoxy	5.6	1.2	0.6	0.26
Boron/5505 Boron/Epoxy	29.6	2.678	0.81	0.23
Spectra® 900/826 Polyethylene/Epoxy	4.45	0.51	0.21	0.32
E-glass/470-36 E-glass/Vinylester	3.54	1.0	0.42	0.32

Generally Orthotropic Lamina



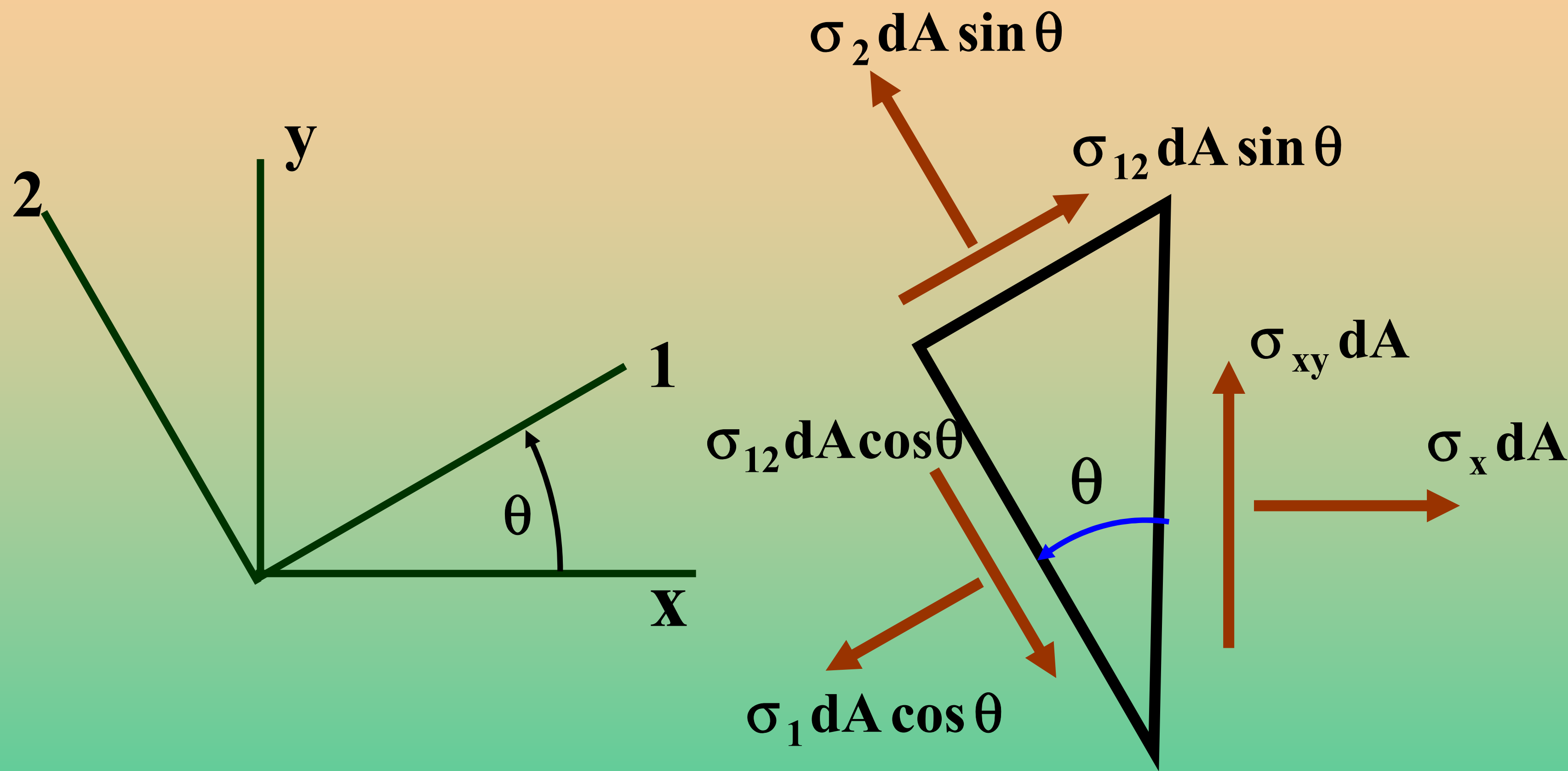
Positive Angle

Generally Orthotropic Lamina



Negative Angle

Stress Element



Equilibrium

$$\sum \mathbf{F_x} = \sigma_{\mathbf{x}} \mathbf{dA} - \sigma_1 \mathbf{dA} \cos^2\theta - \sigma_2 \mathbf{dA} \sin^2\theta$$
$$+ 2\sigma_{12} \mathbf{dA} \sin\theta \cos\theta = 0$$

$$\sum \mathbf{F_y} = \sigma_{\mathbf{xy}} \mathbf{dA} - \sigma_1 \mathbf{dA} \sin\theta \cos\theta + \sigma_2 \mathbf{dA} \sin\theta \cos\theta$$
$$+ \sigma_{12} \mathbf{dA} \left(\sin^2\theta - \cos^2\theta\right) = 0$$

Stress Transformation

$$\sigma_x = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta - 2\sigma_{12} \sin\theta\cos\theta$$

$$\sigma_{xy} = \sigma_1 \sin\theta\cos\theta - \sigma_2 \sin\theta\cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right)$$

Similar derivation for σ_y

Transformation in Matrix Form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}$$

Condensed Matrix Form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$c = \cos\theta \quad \textit{and} \quad s = \sin\theta$$

Transformation Matrix: [T]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} \quad \textit{or} \quad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

Matrices

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix}$$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix}$$

Strain

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

Stress and Strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

Stress and Strain

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix}$$

Reuter's Matrix:

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Lamina Stiffness Matrix

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Stiffness Terms

$$Q_{11} = \frac{S_{22}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{E_1}{\left(1 - \nu_{12}\nu_{21}\right)}$$

$$Q_{22} = \frac{S_{11}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{E_2}{\left(1 - \nu_{12}\nu_{21}\right)}$$

$$Q_{12} = Q_{21} = -\frac{S_{12}}{\left(S_{11}S_{22} - S_{12}^2\right)} = \frac{\nu_{12}E_2}{\left(1 - \nu_{12}\nu_{21}\right)}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1} [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} = [R]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} = [T]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = [R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[R][T][R]^{-1} = [T]^{-T}$$

-T = inverse transpose

General Stress-Strain Behavior

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1}[Q][T]^{-T} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[\overline{Q}] = [T]^{-1}[Q][T]$$

Stress-Strain Behavior

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Explicit Relationships

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

$$\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\cos^2\theta\sin^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

Explicit Relationships

$$\overline{\mathbf{Q}}_{16} = \left(\mathbf{Q}_{11} - \mathbf{Q}_{12} - 2\mathbf{Q}_{66}\right)\cos^3\theta\sin\theta$$
$$+\left(\mathbf{Q}_{22} - \mathbf{Q}_{12} - 2\mathbf{Q}_{66}\right)\cos\theta\sin^3\theta$$
$$\overline{\mathbf{Q}}_{26} = \left(\mathbf{Q}_{11} - \mathbf{Q}_{12} - 2\mathbf{Q}_{66}\right)\cos\theta\sin^3\theta$$
$$+\left(\mathbf{Q}_{22} - \mathbf{Q}_{12} - 2\mathbf{Q}_{66}\right)\cos^3\theta\sin\theta$$
$$\overline{\mathbf{Q}}_{66} = \left(\mathbf{Q}_{11} + \mathbf{Q}_{22} - 2\mathbf{Q}_{12} - 2\mathbf{Q}_{66}\right)\cos\theta^2\sin^2\theta$$
$$+\mathbf{Q}_{66}\left(\cos^4\theta + \sin^4\theta\right)$$

Stress-Strain Behavior

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

Stress-Strain Behavior

$$\left[\bar{\mathbf{S}}\right] = [\mathbf{T}]^{\mathbf{T}}[\mathbf{S}][\mathbf{T}]$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left[\bar{\mathbf{S}}\right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

Inverse Relationship

$$[\overline{\mathbf{Q}}] = [\overline{\mathbf{S}}]^{-1}$$

$$[\overline{\mathbf{S}}] = [\overline{\mathbf{Q}}]^{-1}$$

Symmetric matrices!

Explicit Relationships

$$\bar{S}_{11} = \mathbf{S}_{11}\cos^4\theta + \mathbf{S}_{22}\sin^4\theta + \left(2\mathbf{S}_{12} + \mathbf{S}_{66}\right)\sin^2\theta\cos^2\theta$$

$$\bar{S}_{12} = \left(\mathbf{S}_{11} + \mathbf{S}_{22} - \mathbf{S}_{66}\right)\cos^2\theta\sin^2\theta + \mathbf{S}_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{S}_{22} = \mathbf{S}_{11}\sin^4\theta + \mathbf{S}_{22}\cos^4\theta + \left(2\mathbf{S}_{12} + \mathbf{S}_{66}\right)\sin^2\theta\cos^2\theta$$

Explicit Relationships

$$\overline{\mathbf{S}}_{16} = \left(2\mathbf{S}_{11} - 2\mathbf{S}_{12} - \mathbf{S}_{66}\right)\cos^3\theta\sin\theta$$
$$+\left(2\mathbf{S}_{22} - 2\mathbf{S}_{12} - \mathbf{S}_{66}\right)\cos\theta\sin^3\theta$$
$$\overline{\mathbf{S}}_{26} = \left(2\mathbf{S}_{11} - 2\mathbf{S}_{12} - \mathbf{S}_{66}\right)\cos\theta\sin^3\theta$$
$$+\left(2\mathbf{S}_{22} - 2\mathbf{S}_{12} - \mathbf{S}_{66}\right)\cos^3\theta\sin\theta$$
$$\overline{\mathbf{S}}_{66} = 2\left(2\mathbf{S}_{11} + 2\mathbf{S}_{22} - 4\mathbf{S}_{12} - \mathbf{S}_{66}\right)\cos\theta^2\sin^2\theta$$
$$+\mathbf{S}_{66}\left(\cos^4\theta + \sin^4\theta\right)$$

Engineering Constants

$$\bar{S}_{11} = \frac{1}{E_x} \qquad \bar{S}_{22} = \frac{1}{E_y} \qquad \bar{S}_{66} = \frac{1}{G_{xy}}$$

$$\bar{S}_{12} = -\frac{\nu_{xy}}{E_x} = -\frac{\nu_{yx}}{E_y}$$

$$\bar{S}_{16} = \frac{\eta_{xy,x}}{E_x} = \frac{\eta_{x,xy}}{G_{xy}} \qquad \bar{S}_{26} = \frac{\eta_{xy,y}}{E_y} = \frac{\eta_{y,xy}}{G_{xy}}$$

Engineering Constants

$$\frac{\overline{\mathbf{S}}_{16}}{\overline{\mathbf{S}}_{11}} = \eta_{\mathbf{xy},\mathbf{x}}$$

$$\frac{\overline{\mathbf{S}}_{26}}{\overline{\mathbf{S}}_{66}} = \eta_{\mathbf{y},\mathbf{xy}}$$

Coefficients of Mutual Influence

$\eta_{x,xy}$ *and* $\eta_{y,xy}$

Coefficients of mutual influence of the first kind

Characterize stretching in the x or y direction

caused by shear stress in the xy plane.

$$\eta_{x,xy} = \frac{\varepsilon_x}{\gamma_{xy}} \quad \textit{for} \quad \sigma_{xy} = \tau \quad \textit{all other stresses} = 0$$

Coefficients of Mutual Influence

$\eta_{xy,x}$ *and* $\eta_{xy,y}$

Coefficients of mutual influence of the second kind.

Characterizes shearing in the xy plane caused by normal stress in the xy plane.

$$\eta_{xy,x} = \frac{\gamma_{xy}}{\varepsilon_x} \quad \textit{for} \quad \sigma_x = \sigma \quad \textit{all other stresses} = 0$$

Engineering Constants

$$\frac{1}{\mathbf{E_x}} = \frac{1}{\mathbf{E}_1}\mathbf{c}^4 + \left[\frac{1}{\mathbf{G}_{12}} - \frac{2\nu_{12}}{\mathbf{E}_1}\right]\mathbf{s}^2\mathbf{c}^2 + \frac{1}{\mathbf{E}_2}\mathbf{s}^4$$

$$\frac{1}{\mathbf{E_y}} = \frac{1}{\mathbf{E}_1}\mathbf{s}^4 + \left[\frac{1}{\mathbf{G}_{12}} - \frac{2\nu_{12}}{\mathbf{E}_1}\right]\mathbf{s}^2\mathbf{c}^2 + \frac{1}{\mathbf{E}_2}\mathbf{c}^4$$

Engineering Constants

$$\frac{1}{\nu_{xy}} = \mathbf{E_x}\left[\frac{\nu_{12}}{\mathbf{E}_1}\left(\mathbf{c}^4 + \mathbf{s}^4\right) - \left[\frac{1}{\mathbf{E}_1} + \frac{1}{\mathbf{E}_2} - \frac{1}{\mathbf{G}_{12}}\right]\mathbf{s}^2\mathbf{c}^2\right]$$

$$\frac{1}{\mathbf{G_{xy}}} = 2\left[\frac{2}{\mathbf{E}_1} + \frac{2}{\mathbf{E}_2} + \frac{4\nu_{12}}{\mathbf{E}_1} - \frac{1}{\mathbf{G}_{12}}\right]\mathbf{s}^2\mathbf{c}^2 + \frac{1}{\mathbf{G}_{12}}\left(\mathbf{c}^4 + \mathbf{s}^4\right)$$

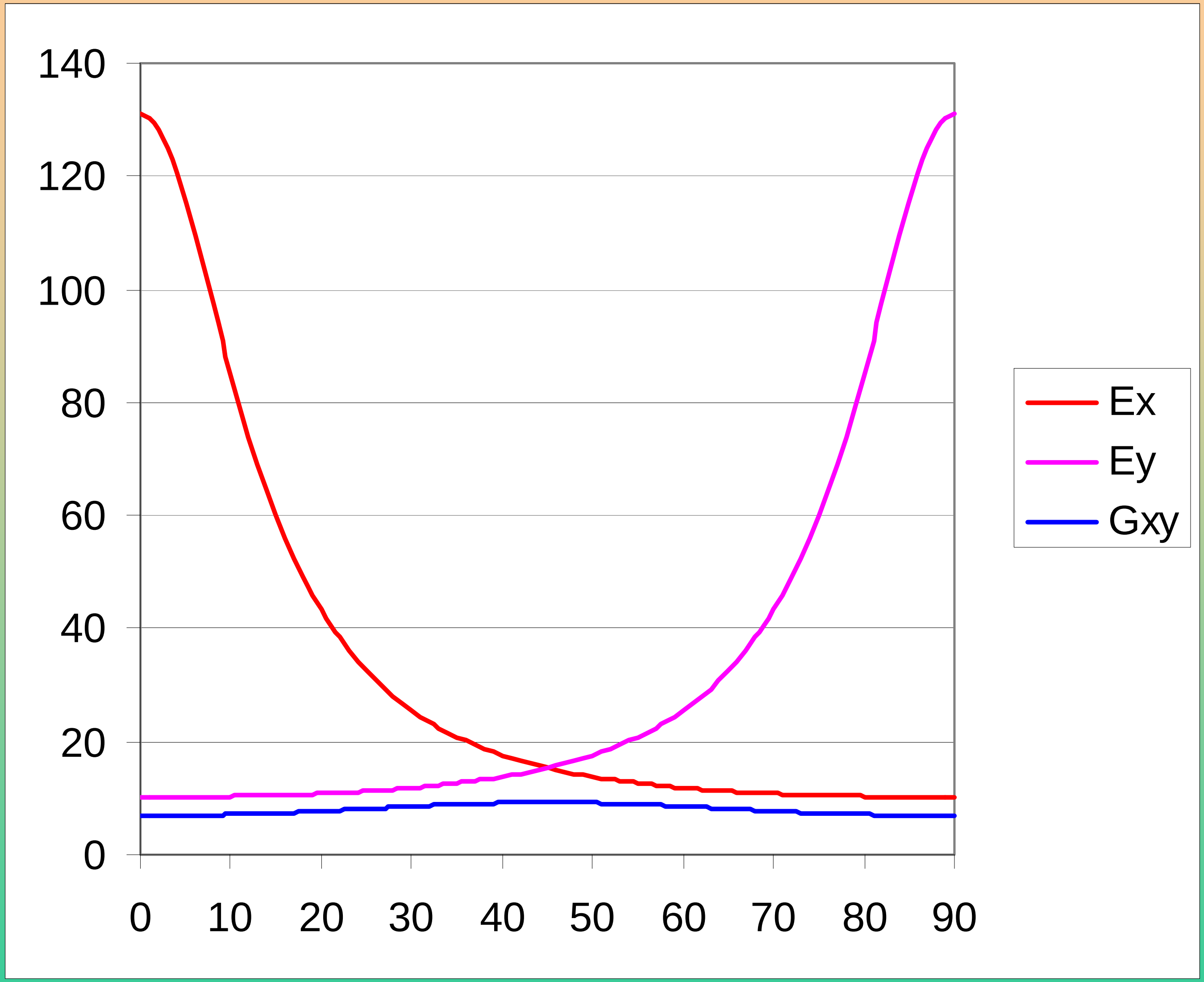
Engineering Constants

$$\eta_{xy,x} = E_x\left[\left[\frac{2}{E_1}+\frac{2\nu_{12}}{E_1}-\frac{1}{G_{12}}\right]sc^3-\left[\frac{2}{E_2}+\frac{2\nu_{12}}{E_1}-\frac{1}{G_{12}}\right]s^3c\right]$$

$$\eta_{xy,y} = E_y\left[\left[\frac{2}{E_1}+\frac{2\nu_{12}}{E_1}-\frac{1}{G_{12}}\right]s^3c-\left[\frac{2}{E_2}+\frac{2\nu_{12}}{E_1}-\frac{1}{G_{12}}\right]sc^3\right]$$

$$\eta_{x,xy} = \frac{G_{xy}}{E_x}\eta_{xy,x} \qquad \eta_{y,xy} = \frac{G_{xy}}{E_y}\eta_{xy,y}$$

T300/934 Graphite/Epoxy



Trig Identities

$$\cos^4\theta = \frac{1}{8}\left(3 + 4\cos 2\theta + \cos 4\theta\right)$$

$$\sin^4\theta = \frac{1}{8}\left(3 - 4\cos 2\theta + \cos 4\theta\right)$$

$$\cos^3\theta\sin\theta = \frac{1}{8}\left(2\sin 2\theta + \sin 4\theta\right)$$

$$\cos\theta\sin^3\theta = \frac{1}{8}\left(2\sin 2\theta - \sin 4\theta\right)$$

$$\cos^2\theta\sin^2\theta = \frac{1}{8}\left(1 - \cos 4\theta\right)$$

Alternate Form for Stiffness

$$\overline{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$\overline{Q}_{12} = U_4 - U_3 \cos 4\theta$$

$$\overline{Q}_{22} = U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$\overline{Q}_{16} = \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta$$

$$\overline{Q}_{26} = \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta$$

$$\overline{Q}_{66} = \frac{1}{2}\left(U_1 - U_4\right) - U_3 \cos 4\theta$$

Invariants

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

$$U_4 = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

Alternate Form for Compliances

$$\bar{S}_{11} = V_1 + V_2 \cos 2\theta + V_3 \cos 4\theta$$

$$\bar{S}_{12} = V_4 - V_3 \cos 4\theta$$

$$\bar{S}_{22} = V_1 - V_2 \cos 2\theta + V_3 \cos 4\theta$$

$$\bar{S}_{16} = V_2 \sin 2\theta + 2V_3 \sin 4\theta$$

$$\bar{S}_{26} = V_2 \sin 2\theta - 2V_3 \sin 4\theta$$

$$\bar{S}_{66} = 2\left(V_1 - V_4\right) - 4V_3 \cos 4\theta$$

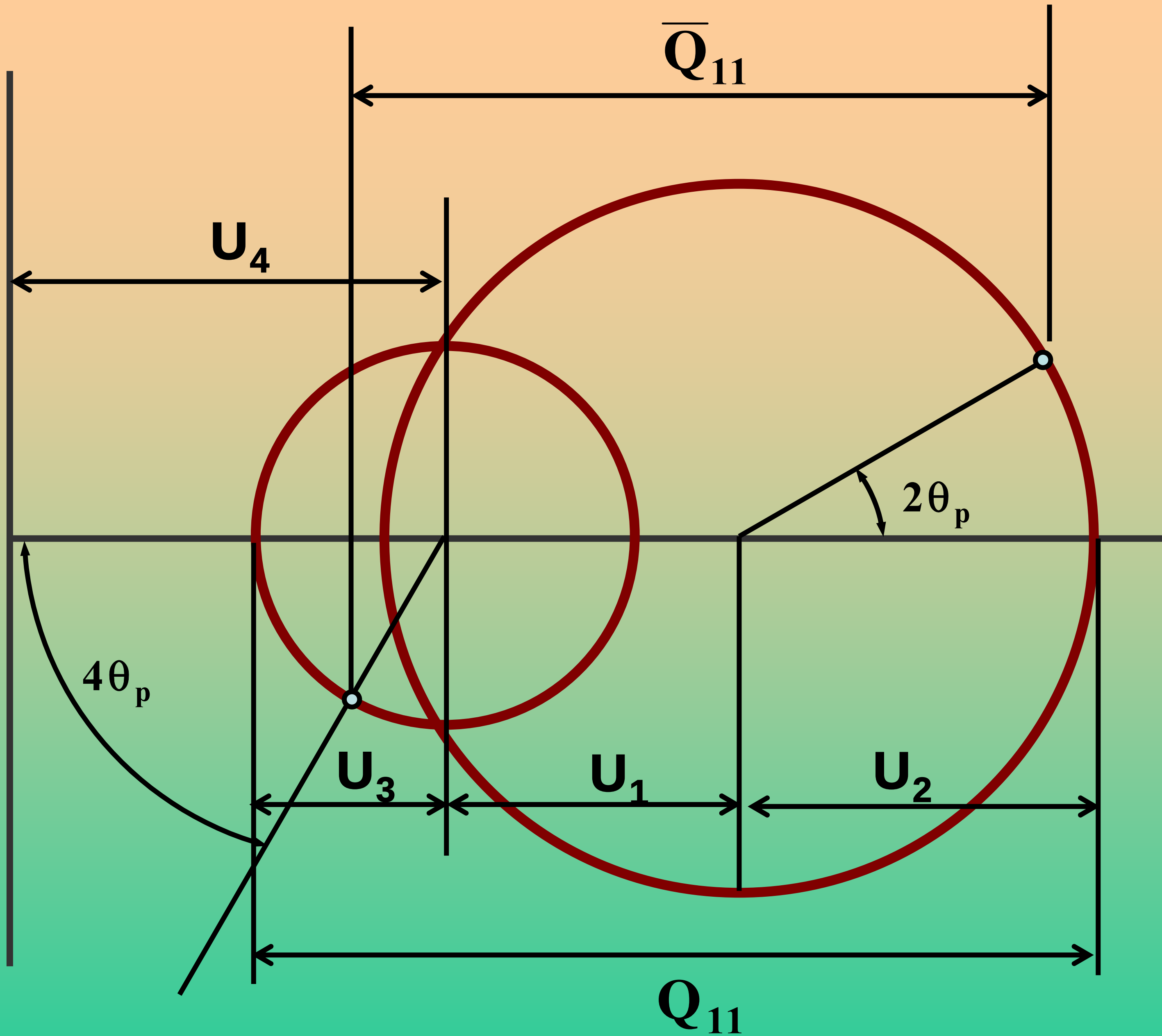
Invariants

$$V_1 = \frac{1}{8}\left(3S_{11} + 3S_{22} + 2S_{12} + S_{66}\right)$$

$$V_2 = \frac{1}{2}\left(S_{11} - S_{22}\right)$$

$$V_3 = \frac{1}{8}\left(S_{11} + S_{22} - 2S_{12} - S_{66}\right)$$

$$V_4 = \frac{1}{8}\left(S_{11} + S_{22} + 6S_{12} - S_{66}\right)$$



$\overline{Q}_{11}$
U_4
$2\theta_p$
$4\theta_p$
U_3
U_1
U_2
Q_{11}

Transversely Isotropic Material

$$E_2 = E_3$$

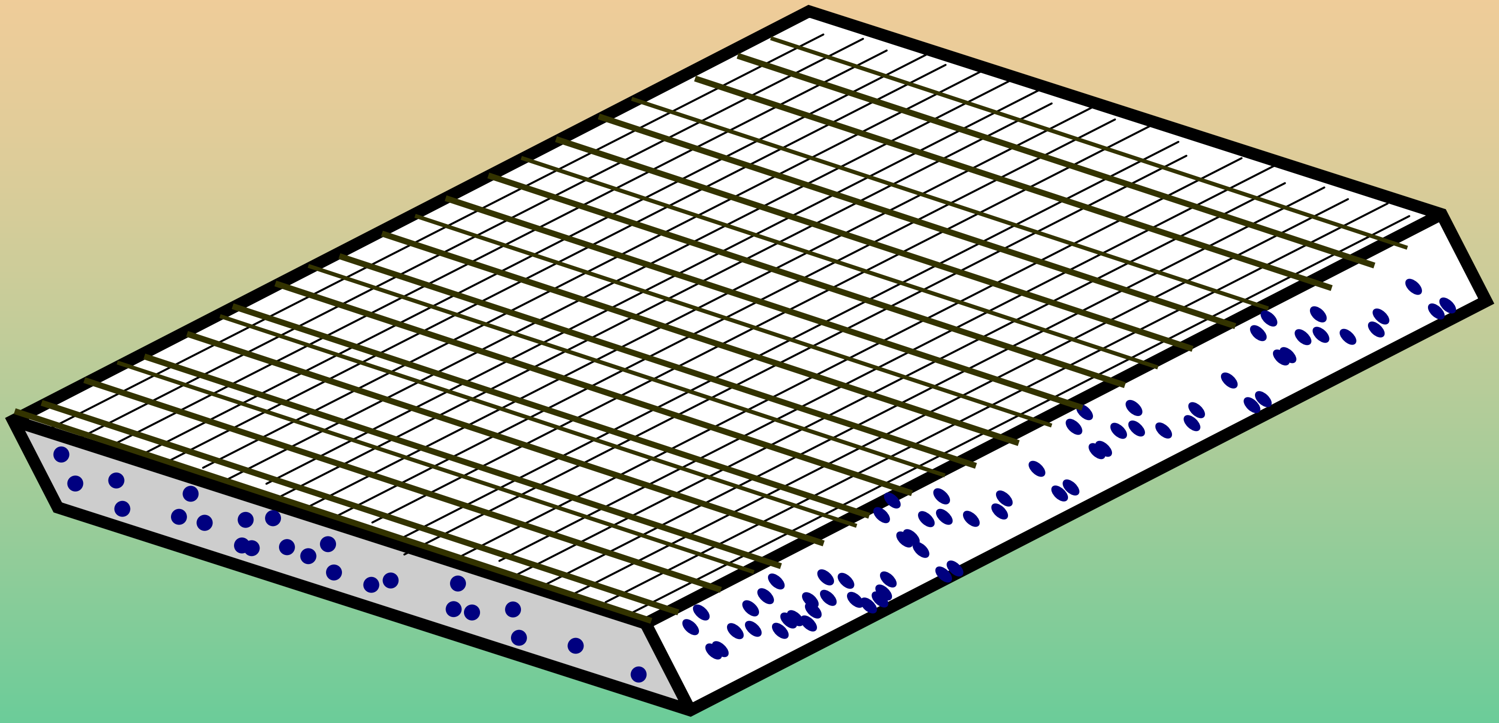
$$G_{12} = G_{13}$$

$$\nu_{21} = \nu_{31}$$

$$\nu_{23} = \nu_{32}$$

$$G_{23} = \frac{E_2}{2\left(1 + \nu_{32}\right)}$$

Balanced Orthotropic Material



Balanced Orthotropic Lamina

1. 0^o and 90^o cross-plies
2. Woven Materials
3. 3 Components

$$E_1 = E_2$$

$$Q_{11} = Q_{22}$$

$$S_{11} = S_{22}$$