8.4 Linearity

A function $f(x)$ is a linear function of the independent variable $x$ if, and only if, it satisfies two properties.

1. *Additivity (or superposition)*
   \[
   f(x_1 + x_2) = f(x_1) + f(x_2)
   \]
   for all $x_1$ and $x_2$ in the domain of $f(x)$.

2. *Homogeneity*
   \[
   f(\alpha x) = \alpha f(x)
   \]
   for all $x$ in the domain of $f(x)$ and all scalars $\alpha$.

**Examples of static linear systems**

1. $f(x) = 2x, \quad x \in \mathbb{R}$ (All real numbers)
   \[
   f(x_1 + x_2) = 2(x_1 + x_2) = 2x_1 + 2x_2
   \]
   \[
   f(x_1) + f(x_2) = 2x_1 + 2x_2
   \]
   So,
   \[
   f(x_1 + x_2) = f(x_1) + f(x_2) \quad \rightarrow \quad \text{satisfies additivity}
   \]
   Now consider
   \[
   f(\alpha x) = 2(\alpha x) = 2\alpha x
   \]
   \[
   \alpha f(x) = \alpha(2x) = 2\alpha x
   \]
   So,
   \[
   f(\alpha x) = \alpha f(x) \quad \rightarrow \quad \text{satisfies homogeneity}
   \]
   Thus, the system is linear.

2. $f(x) = 2x + 9, \quad x \in \mathbb{R}$
   \[
   f(x_1 + x_2) = 2(x_1 + x_2) + 9 = 2x_1 + 2x_2 + 9
   \]
   \[
   f(x_1) + f(x_2) = (2x_1 + 9) + (2x_2 + 9) = 2x_1 + 2x_2 + 18
   \]
So,
\[ f(x_1 + x_2) \neq f(x_1) + f(x_2) \quad \rightarrow \quad \text{does not satisfy additivity} \]
Consider,
\[
\begin{align*}
  f(ax) &= 2(ax) + 9 \\
  \alpha f(x) &= \alpha(2x + 9) = 2ax + 9\alpha
\end{align*}
\]
So,
\[ f(ax) \neq \alpha f(x) \quad \rightarrow \quad \text{does not satisfy homogeneity} \]
Thus, the system is not linear.

3. \( f(x) = x^2 + x \)

\[
\begin{align*}
  f(x_1 + x_2) &= (x_1 + x_2)^2 + (x_1 + x_2) = x_1^2 + x_2^2 + 2x_1x_2 + x_1 + x_2 \\
  f(x_1) + f(x_2) &= x_1^2 + x_1 + x_2^2 + x_2 = x_1^2 + x_2^2 + x_1 + x_2
\end{align*}
\]
So,
\[ f(x_1 + x_2) \neq f(x_1) + f(x_2) \quad \rightarrow \quad \text{does not satisfy additivity} \]
Consider,
\[
\begin{align*}
  f(ax) &= (ax)^2 + (ax) = \alpha^2x^2 + \alpha x \\
  \alpha f(x) &= \alpha(x^2 + x) = \alpha x^2 + \alpha x
\end{align*}
\]
So,
\[ f(ax) \neq \alpha f(x) \quad \rightarrow \quad \text{does not satisfy homogeneity} \]
Thus, the system is not linear.

4. \( f(x) = 3x, \quad x \in \{0, 2, 4, 6, \ldots \} \)

\[
\begin{align*}
  f(x_1 + x_2) &= 3(x_1 + x_2) = 3x_1 + 3x_2 \\
  f(x_1) + f(x_2) &= 3x_1 + 3x_2
\end{align*}
\]
So,
\[ f(x_1 + x_2) = f(x_1) + f(x_2) \quad \rightarrow \quad \text{satisfies additivity} \]
Now, consider,
\[ f(ax) \]
This function makes sense only if \( ax \in \{0, 2, 4, \ldots \} \). But this will not be the case for all scalar values of \( \alpha \). For example, if we consider \( \alpha = 1.2 \) or \( \alpha = \sqrt{3} \) or \( \alpha = -2 \), then \( f(ax) \) is undefined. Hence, the function is not homogeneous.
So, the function \( f(.) \) is additive but not homogeneous and thus, the system is not linear.
5. \( f(x) = 2x \)
where,
\[
x = \begin{bmatrix} a \\ 0 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} 0 \\ b \end{bmatrix}
\]
where, \( a, b \in \mathbb{R} \)

For example, we can have
\[
x = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}
\]
but we cannot have
\[
x = \begin{bmatrix} 1.2 \\ 2 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} -2 \\ 6 \end{bmatrix}
\]
Using "\(^T\)" to denote transpose
\[
f(x_1 + x_2) = f([a \\ 0]') + [0 \\ b]' = f(a \\ b)
\]
which is undefined for non-zero values of \( a \) and \( b \). On the other hand,
\[
f(x_1) + f(x_2) = 2[a \\ 0]' + 2[0 \\ b]' = [2a \\ 2b]'
\]
So,
\[
f(x_1 + x_2) \neq f(x_1) + f(x_2) \rightarrow \text{(does not satisfy additivity)}
\]
Consider,
\[
f(\alpha x) = f(\alpha[a \\ 0]') = f([\alpha a \\ 0]') = 2[\alpha a \\ 0]' = [2\alpha a \\ 0]'
\]
\[
\alpha f(x) = \alpha f([a \\ 0]') = \alpha(2[a \\ 0]') = \alpha[2a \\ 0]' = [2\alpha a \\ 0]'
\]
or
\[
f(\alpha x) = f(\alpha[0 \\ b]') = f([0 \\ \alpha b]') = 2[0 \\ \alpha b]' = [0 \\ 2\alpha b]'
\]
\[
\alpha f(x) = \alpha f([0 \\ b]') = \alpha(2[0 \\ b]') = \alpha[0 \\ 2b]' = [0 \\ 2\alpha b]'
\]
In both the above cases,
\[
f(\alpha x) = \alpha f(x) \rightarrow \text{(satisfies homogeneity)}
\]
Hence, the function \( f(\cdot) \) is not additive, but it is homogeneous, and so the system is not linear.

All the above examples were for static systems that did not have the time element in them. But, our main concern is with dynamical systems where time response plays an important role. How do we define and test for linearity of dynamical systems?
Consider Figure 8.9.

The input $u$ is a function of time and so is the output $y$. So,

$$f(u) = y$$

It is important to understand the above function clearly. Unlike in the static case, $u$ does not have a single value. It represents, an input signal over some interval in time (which could be an infinite interval too). The output $y$ is also a function of time and represents the output signal over some interval of time. These time intervals are entirely arbitrary. For example, we can apply the input signal from 2 seconds to 144 seconds and measure the output signal from 0 seconds to 100 seconds.

*Additivity* implies that,

$$f(u_1 + u_2) = f(u_1) + f(u_2)$$

The left hand side represents the output of the system when the two input signals are applied together. The right hand side represents the sum of the two output signals obtained separately by applying the two input signals separately. If these are identical over any intervals of time then the system satisfies the property of additivity.

This is shown in Figure 8.10.

*Homogeneity* implies that,

$$f(\alpha u) = \alpha f(u)$$

The left hand side represents the output of the system when the input signal $u$ is scaled up (or down) by a factor $\alpha$ and applied. The right hand side is the output signal, scaled up by $\alpha$, when the input signal is $u$. If these are identical over any intervals of time then the system satisfies the property of homogeneity.

This is shown in Figure 8.11.

So, a dynamical system that satisfies both these properties is a linear dynamical system.

A word of caution: *Do not look for linearity in the time domain.*
Figure 8.10: Response of an additive dynamical system

Figure 8.11: Response of a homogeneous dynamical system
The dynamical systems we consider are those that are modelled by linear, time-invariant, ordinary differential equations.

For example, consider the following differential equation that models our system.

\[ a_0 y(t) + a_1 \frac{dy(t)}{dt} + a_2 \frac{d^2 y(t)}{dt^2} + \cdots + a_n \frac{d^n y(t)}{dt^n} = b_0 u(t) + b_1 \frac{du(t)}{dt} + b_2 \frac{d^2 u(t)}{dt^2} + \cdots + b_m \frac{d^m u(t)}{dt^m} \quad (8.1) \]

where, \( u \) and \( y \) are the input and output of the system, and \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_m \) are constants.

To show that the above equation represents a linear system, we check the two properties as follows:

To prove additivity, let \( u_1 \) and \( u_2 \) be the inputs to the system and let the corresponding outputs be \( y_1 \) and \( y_2 \). So, each of the input-output pair of functions \((u, y_1)\) and \((u, y_2)\) should satisfy Eqn. \((8.1)\). So,

\[ a_0 y_1(t) + a_1 \frac{dy_1(t)}{dt} + a_2 \frac{d^2 y_1(t)}{dt^2} + \cdots + a_n \frac{d^n y_1(t)}{dt^n} = b_0 u_1(t) + b_1 \frac{du_1(t)}{dt} + b_2 \frac{d^2 u_1(t)}{dt^2} + \cdots + b_m \frac{d^m u_1(t)}{dt^m} \quad (8.2) \]

\[ a_0 y_2(t) + a_1 \frac{dy_2(t)}{dt} + a_2 \frac{d^2 y_2(t)}{dt^2} + \cdots + a_n \frac{d^n y_2(t)}{dt^n} = b_0 u_2(t) + b_1 \frac{du_2(t)}{dt} + b_2 \frac{d^2 u_2(t)}{dt^2} + \cdots + b_m \frac{d^m u_2(t)}{dt^m} \quad (8.3) \]

Now, assume that we apply the input \((u_1 + u_2)\) to the system and let the corresponding output be \( \hat{y} \). Then, the input-output pair \((u_1 + u_2, \hat{y})\) should also satisfy Eqn. \((8.1)\). And so,

\[ a_0 \hat{y}(t) + a_1 \frac{d\hat{y}(t)}{dt} + a_2 \frac{d^2 \hat{y}(t)}{dt^2} + \cdots + a_n \frac{d^n \hat{y}(t)}{dt^n} = b_0 [u_1(t) + u_2(t)] + b_1 \frac{d[u_1(t) + u_2(t)]}{dt} + b_2 \frac{d^2 [u_1(t) + u_2(t)]}{dt^2} + \cdots + b_m \frac{d^m [u_1(t) + u_2(t)]}{dt^m} \quad (8.4) \]

If we can show that \( \hat{y}(t) = y_1(t) + y_2(t) \), then additivity is proved.

We can write \((8.4)\) as,

\[ a_0 \hat{y}(t) + a_1 \frac{d\hat{y}(t)}{dt} + a_2 \frac{d^2 \hat{y}(t)}{dt^2} + \cdots + a_n \frac{d^n \hat{y}(t)}{dt^n} = \left[ b_0 u_1(t) + b_1 \frac{du_1(t)}{dt} + b_2 \frac{d^2 u_1(t)}{dt^2} + \cdots + b_m \frac{d^m u_1(t)}{dt^m} \right] + \left[ b_0 u_2(t) + b_1 \frac{du_2(t)}{dt} + b_2 \frac{d^2 u_2(t)}{dt^2} + \cdots + b_m \frac{d^m u_2(t)}{dt^m} \right] \quad (8.5) \]
On substituting from (8.2) and (8.3), this becomes,

\[
\begin{align*}
& a_0 \dot{y}(t) + a_1 \frac{d\dot{y}(t)}{dt} + a_2 \frac{d^2\dot{y}(t)}{dt^2} + \cdots + a_n \frac{d^n\dot{y}(t)}{dt^n} \\
= & \left[ a_0 y_1(t) + a_1 \frac{dy_1(t)}{dt} + a_2 \frac{d^2y_1(t)}{dt^2} + \cdots + a_n \frac{d^n y_1(t)}{dt^n} \right] \\
& + \left[ a_0 y_2(t) + a_1 \frac{dy_2(t)}{dt} + a_2 \frac{d^2y_2(t)}{dt^2} + \cdots + a_n \frac{d^n y_2(t)}{dt^n} \right] \\
= & a_0 [y_1(t) + y_2(t)] + a_1 \frac{dy_1(t) + y_2(t)}{dt} + a_2 \frac{d^2[y_1(t) + y_2(t)]}{dt^2} + \\
& \cdots + a_n \frac{d^n[y_1(t) + y_2(t)]}{dt^n} \\
\end{align*}
\]

which is sufficient to show that \( \dot{y}(t) = y_1(t) + y_2(t) \). This proves additivity.

To prove homogeneity, let \((u, y)\) be the input-output pair that satisfies (8.1). Now, assume that the input to the system is \(\alpha u\) and the corresponding output is \(\dot{y}\). Then \((\alpha u, \dot{y})\) should satisfy,

\[
\begin{align*}
& a_0 \dot{\dot{y}}(t) + a_1 \frac{d\dot{\dot{y}}(t)}{dt} + a_2 \frac{d^2\dot{\dot{y}}(t)}{dt^2} + \cdots + a_n \frac{d^n\dot{\dot{y}}(t)}{dt^n} \\
= & b_0 [\alpha u(t)] + b_1 \frac{d[\alpha u(t)]}{dt} + b_2 \frac{d^2[\alpha u(t)]}{dt^2} + \cdots + b_m \frac{d^m[\alpha u(t)]}{dt^m} \\
\end{align*}
\]

If we can show that \( \dot{y} = \alpha y \), then homogeneity is proved. Eqn (8.7) can be rearranged and then, using (8.1), it can be written as

\[
\begin{align*}
& a_0 \dot{\dot{y}}(t) + a_1 \frac{d\dot{\dot{y}}(t)}{dt} + a_2 \frac{d^2\dot{\dot{y}}(t)}{dt^2} + \cdots + a_n \frac{d^n\dot{\dot{y}}(t)}{dt^n} \\
= & \alpha \left[ b_0 u(t) + b_1 \frac{du(t)}{dt} + b_2 \frac{d^2 u(t)}{dt^2} + \cdots + b_m \frac{d^m u(t)}{dt^m} \right] \\
= & \alpha \left[ a_0 y(t) + a_1 \frac{dy(t)}{dt} + a_2 \frac{d^2y(t)}{dt^2} + \cdots + a_n \frac{d^n y(t)}{dt^n} \right] \\
= & a_0 [\alpha y(t)] + a_1 \frac{d[\alpha y(t)]}{dt} + a_2 \frac{d^2[\alpha y(t)]}{dt^2} + \cdots + a_n \frac{d^n[\alpha y(t)]}{dt^n} \\
\end{align*}
\]

which is sufficient to prove that \( \dot{y} = \alpha y \). This proves homogeneity of the system. Thus, the system represented by the ordinary differential equation (8.1) is linear.

**PROBLEM SET 1**

1. Determine which of these functions are linear:
   
   (a) \( f(x) = -ax + b \) where \( a > 0, b > 0 \) are constants.
(b) \( f(x) = 3x \) where \( x = \{(1,1,a)^T : a \in \mathcal{R}\} \)
Note that the symbol 'T' stands for matrix transpose. So \((1, 1, a)^T\) is a column vector. Also, \(\mathcal{R}\) is the set of real numbers.

(c) \( f(x) = cx \) where \( c > 0 \) and \( x = \{(a,b,0)^T : a \in \mathcal{R}, b \in \mathcal{R}\} \cup \{(a,0,b)^T : a \in \mathcal{R}, b \in \mathcal{R}\} \)

2. Determine which of the following systems, represented by ordinary differential equations, are linear:

(a) \(3 \frac{d^2 y}{dt^2} + \frac{dy}{dt} + 5y = 9 \frac{du}{dt} + u\)
(b) \(-7 \frac{d^3 y}{dt^3} + 3 \frac{dy}{dt} + 2y = 5 \frac{du}{dt} + 7u\)
(c) \(8 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = 3 \frac{du}{dt} + u + 3\)
(d) \(8 \frac{d^2 y}{dt^2} + 2t \frac{dy}{dt} = 3 \frac{du}{dt} + 4u\)
(e) \(8y \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y = 3 \frac{du}{dt} + u\)
(f) \(7 \frac{dy}{dt} + 3y = 9\)

3. Consider a system which takes as its input any piecewise continuous signal and produces an output at time \(t\) as the algebraic sum of the magnitude of the discontinuous jumps the input signal has undergone till time \(t\). Is this system linear?

Note that a piecewise continuous signal is one which has only a finite number of discontinuities in any given finite time interval and has finite magnitude. For example, a step signal is piecewise continuous. So is a signal with multiple but finite number of steps. So is a sinusoidal signal, and so on. In fact, almost any signal that you can plot on a sheet of paper has this property.

8.5 Causal System or Causality

A causal system is one in which the output at time \(t\) depends only on the input signal up to time \(t\), and not on the input after time \(t\). This is intuitively obvious but needs to be stated, since mathematical models can be quite tricky, especially when you are integrating time dependent functions to obtain responses.